The facial weak order in hyperplane arrangements

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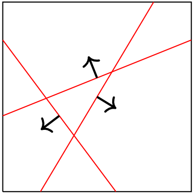
Outline

- Arranging hyperplanes.
- The facial weak order and its ~~1~~, ~~2~~, ~~3~~, 4 (!) definitions.
- Yeah, but is it a lattice?
- Some other properties.

Hyperplanes

- $(V, \langle\cdot,\cdot\rangle)$ - n-dim real Euclidean vector space.
- A *hyperplane H* is codim 1 subspace of V with normal e_H.

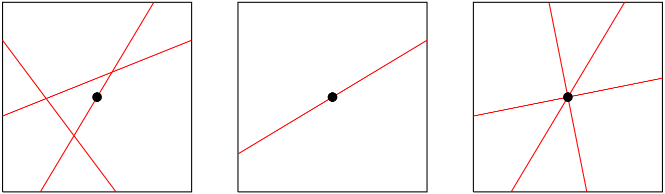
Example



Arrangements

- A *hyperplane arrangement* is $\mathcal{A} = \{H_1, H_2, \ldots, H_k\}$.
- $\mathcal{A}$ is *central* if $\{0\} \subseteq \bigcap \mathcal{A}$.
- $\mathcal{A}$ is *essential* if $\operatorname{span}\{e_H\}_{H \in \mathcal{A}} = V$.
- $\mathcal{A}$ central & essential $\Rightarrow \{0\} = \bigcap \mathcal{A}$.

Example



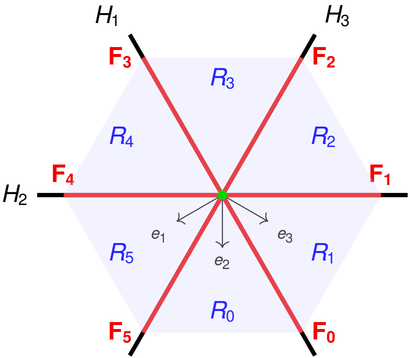
Not central
Essential

Central
Not essential

Central
Essential

Regions and faces

- *Regions* $\mathscr{R}_{\mathcal{A}}$ - connected components of V without $\mathcal{A}$.
- *Faces* $\mathscr{F}_{\mathcal{A}}$ - intersections of closures of some regions.



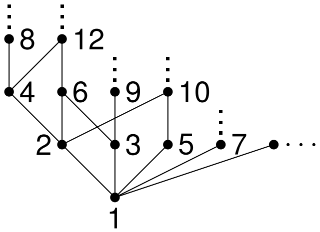
Lattice

- *Lattice* - poset where every two elements have a *meet* (greatest lower bound) and *join* (least upper bound).

Example

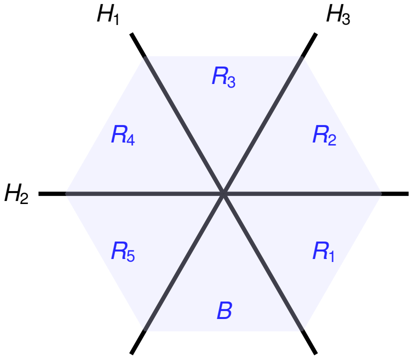
The lattice $(\mathbb{N}, |)$ where $a \leq b \iff a \mid b$.

- meet - greatest common divisor
- join - least common multiple



Poset of regions

- *Base region* $B \in \mathscr{R}_{\mathcal{A}}$ - some fixed region
- *Separation set for* $R \in \mathscr{R}_{\mathcal{A}}$
  $S(R) := \{H \in \mathcal{A} \mid H \text{ separates } R \text{ from } B\}$



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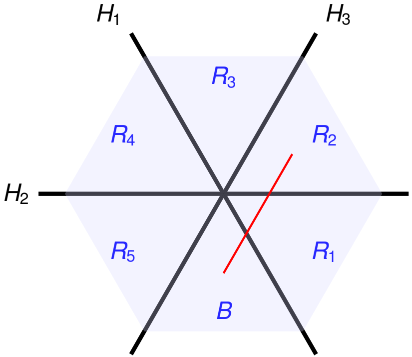

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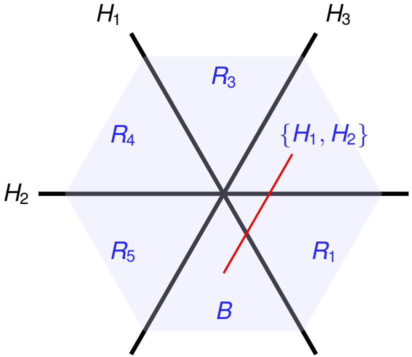

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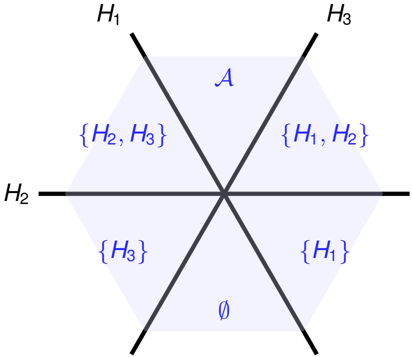

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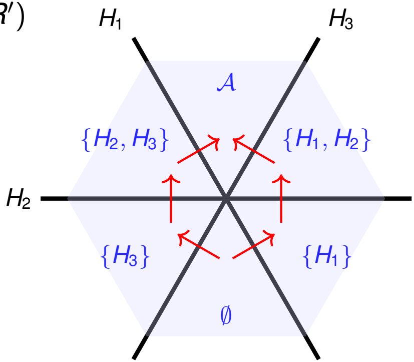
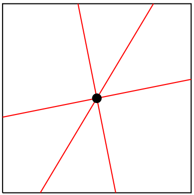
  $S(R) := \{H \in \mathcal{A} \mid H \text{ separates } R \text{ from } B\}$
- *Poset of regions* $\mathrm{PR}(\mathcal{A}, B)$ where
  $R \leq_{\mathrm{PR}} R' \iff S(R) \subseteq S(R')$



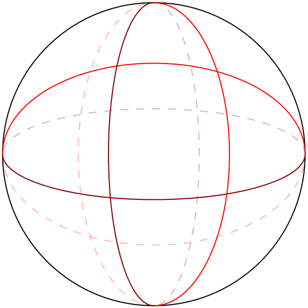
Poset of regions

- A region R is *simplicial* if normal vectors for boundary hyperplanes are linearly independent.
- $\mathcal{A}$ is *simplicial* if all $\mathscr{R}_{\mathcal{A}}$ simplicial.

Example



Simplicial



Not simplicial

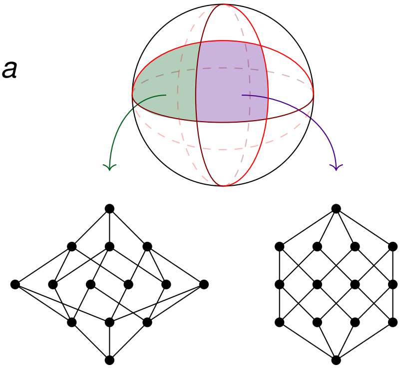
Lattice of regions

An arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is *simplicial* if every region is simplicial (*i.e.*, has n boundary hyperplanes).

Theorem (Björner, Edelman, Ziegler '90)

If $\mathcal{A}$ is simplicial then $\mathrm{PR}(\mathcal{A}, B)$ is a lattice for any $B \in \mathscr{R}_{\mathcal{A}}$.

If $\mathrm{PR}(\mathcal{A}, B)$ is a lattice then B is simplicial.



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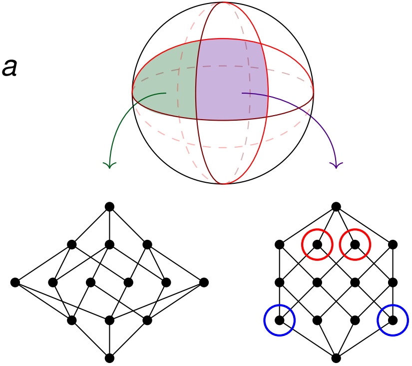

Coxeter arrangements

Example

A *Coxeter arrangement* is the hyerplane arrangement associated to a Coxeter group.

Coxeter Groups		**Hyperplane Arrangements**
Reflecting hyperplanes	$\leftrightarrow$	Hyperplane arrangement
Root system	$\leftrightarrow$	Normals to hyperplanes
Inversion sets	$\leftrightarrow$	Seperation sets
Weak order	$\leftrightarrow$	Poset of regions

Motivation

- **2001:** Krob, Latapy, Novelli, Phan, and Schwer extended the weak order of type A Coxeter groups to all the faces of its associated arrangement.
- **2006:** Palacios and Ronco extended this new order to Coxeter groups of all types using cover relations.
- **2016:** D, Hohlweg and Pilaud gave a global equivalent to this extension and showed it's a lattice.

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- **2001:** Krob, Latapy, Novelli, Phan, and Schwer extended the weak order of type A Coxeter groups to all the faces of its associated arrangement.
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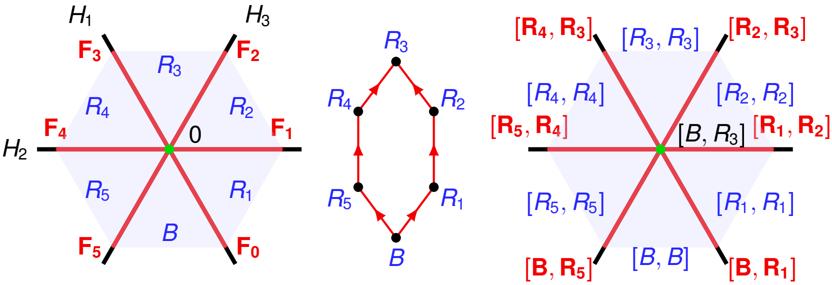
- **2016:** D, Hohlweg and Pilaud gave a global equivalent to this extension and showed it's a lattice.
- Questions: Can we extend this to hyperplane arrangements? Can we find both local and global definitions? When do we actually get a lattice?

Facial intervals

Proposition (Björner, Las Vergas, Sturmfels, White, Ziegler '93)

Let $\mathcal{A}$ be central with base region B. For every $F \in \mathscr{F}_{\mathcal{A}}$ there is a unique interval $[m_F, M_F]$ in $\mathrm{PR}(\mathcal{A}, B)$ such that
$$[m_F, M_F] = \left\{ R \in \mathscr{R}_{\mathcal{A}} \;\middle|\; F \subseteq \overline{R} \right\}$$



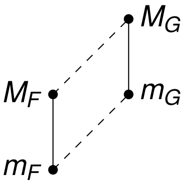
Facial weak order

Let $\mathcal{A}$ be a central hyperplane arrangement and B a base region in $\mathscr{R}_{\mathcal{A}}$.

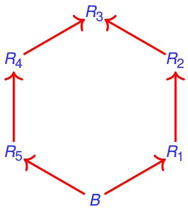
Definition

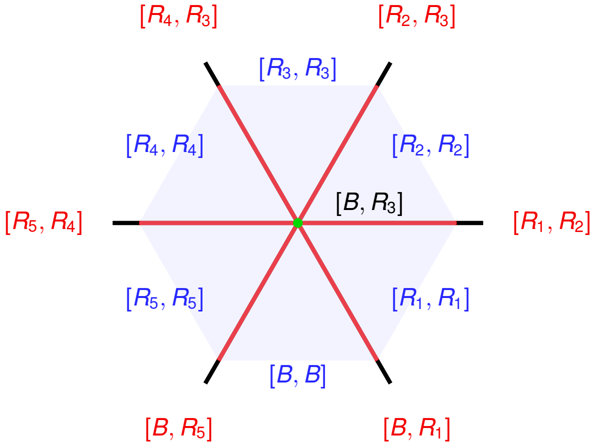
The *facial weak order* is the order $\mathrm{FW}(\mathcal{A}, B)$ on $\mathscr{F}_{\mathcal{A}}$ where for $F, G \in \mathscr{F}_{\mathcal{A}}$:

$$F \leq G \iff m_F \leq_{\mathrm{PR}} m_G \text{ and } M_F \leq_{\mathrm{PR}} M_G$$



Facial weak order - Example





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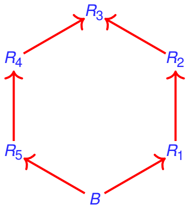

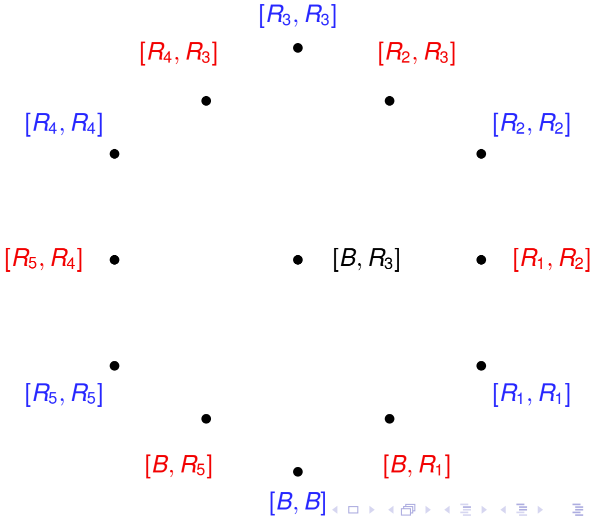

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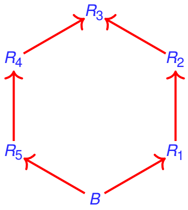
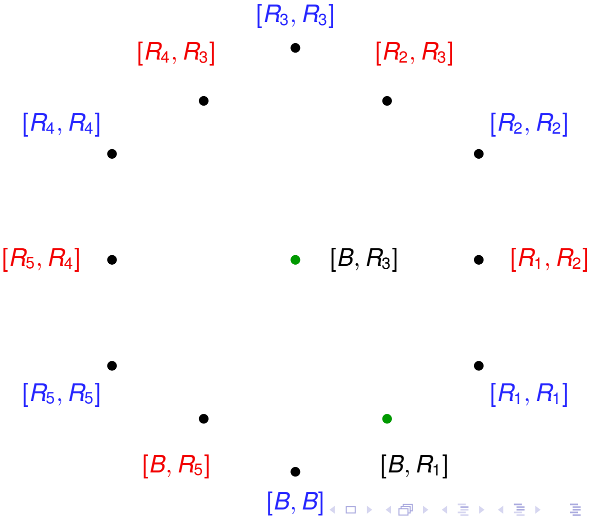
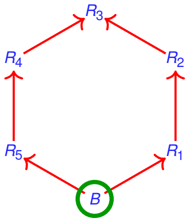
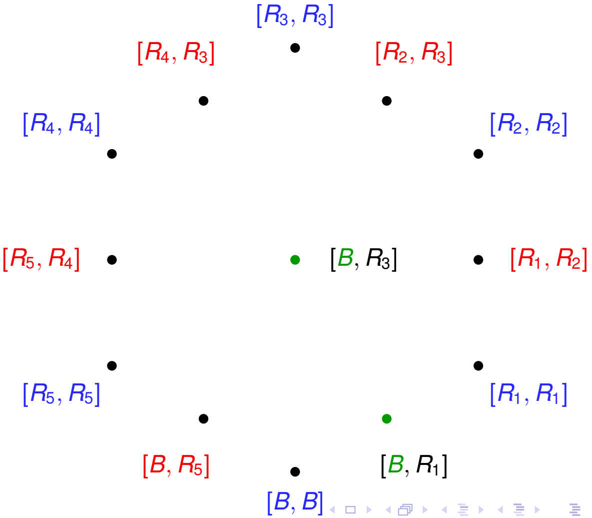




Facial weak order - Example





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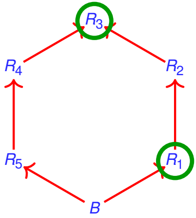

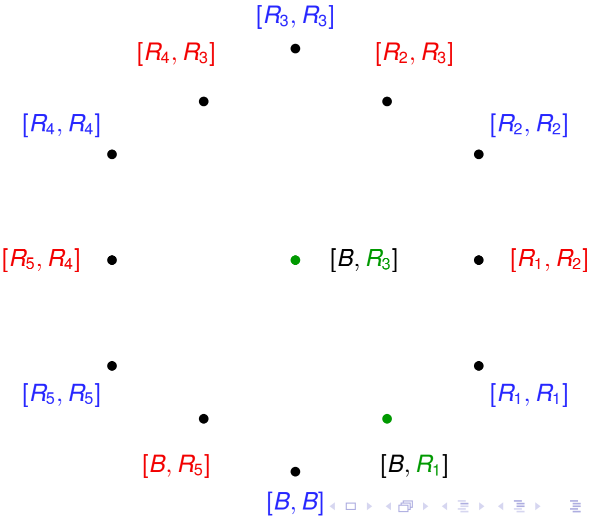

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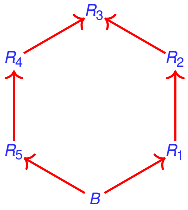

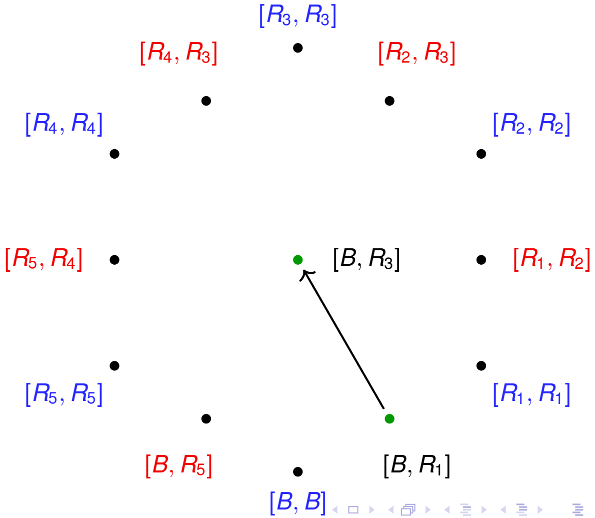

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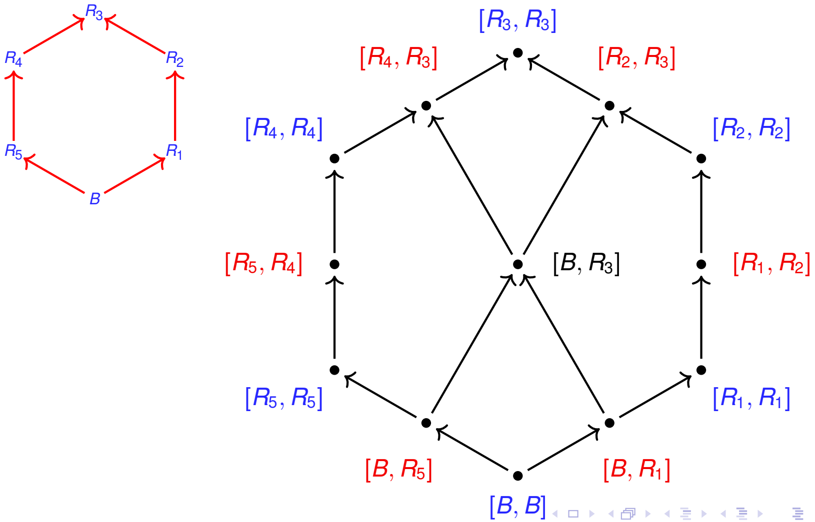


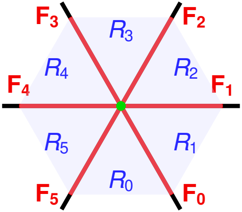
Cover relations

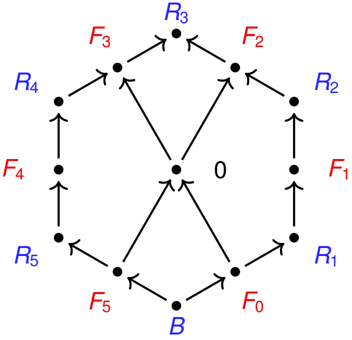
Proposition (D., Hohlweg, McConville, Pilaud, '19+)

For $F, G \in \mathscr{F}_{\mathcal{A}}$ *if* $|\dim(F) - \dim(G)| = 1$ *and*

1. $F \subseteq G$, $M_F = M_G$, *or*
2. $G \subseteq F$, $m_F = m_G$.

then $F \lessdot G$.



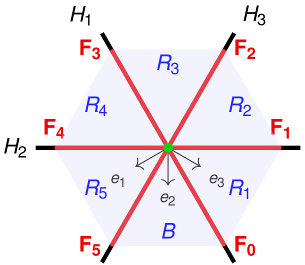


Covectors

- *covector* - a sign vector in $\{-, 0, +\}^{\mathcal{A}}$ with signs relative to hyperplanes.
- $\mathcal{L} \subseteq \{-, 0, +\}^{\mathcal{A}}$ - set of covectors

Example

$$F_4(H_1) = +;\ F_4(H_2) = 0;\ F_4(H_3) = - \qquad F_4 \leftrightarrow (+, 0, -)$$



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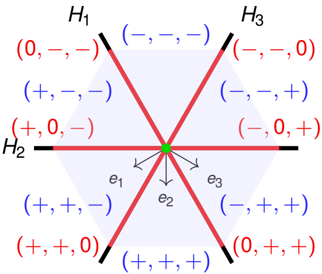


Covector operations

For $X, Y \in \mathcal{L} \subseteq \{-, 0, +\}^{\mathcal{A}}$

- *Composition:* $(X \circ Y)(H) = \begin{cases} Y(H) & \text{if } X(H) = 0 \\ X(H) & \text{otherwise} \end{cases}$
- *Reorientation:* $(X_{-Y})(H) = \begin{cases} -X(H) & \text{if } Y(H) = 0 \\ X(H) & \text{otherwise} \end{cases}$

$\star$ For $F \in \mathscr{F}_{\mathcal{A}}$, $[m_F, M_F] = [F \circ B, F \circ -B]$

Example

Let $\mathcal{A} = \{H_1, H_2, H_3, H_4, H_5\}$.

$$X = (-, 0, +, +, 0) \qquad Y = (0, 0, -, 0, +)$$

Then

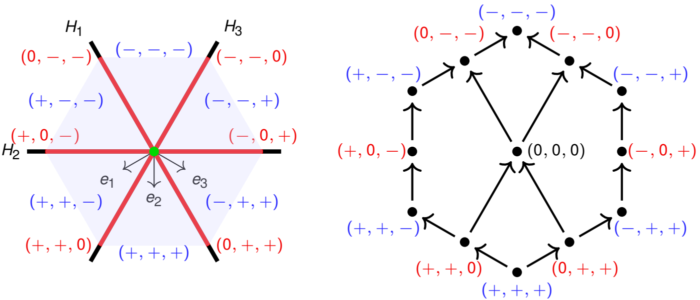
$$X \circ Y = (-, 0, +, +, +) \qquad X_{-Y} = (+, 0, +, -, 0)$$

Covector definition

Definition

For $X, Y \in \mathcal{L}$:

$$X \leq_{\mathcal{L}} Y \iff X(H) \geq Y(H) \quad \forall H \text{ with } - < 0 < +$$



Zonotopes

- *Zonotope $Z_{\mathcal{A}}$* is the convex polytope:

$$Z_{\mathcal{A}} := \left\{ v \in V \;\middle|\; v = \sum_{i=1}^{k} \lambda_i e_i, \text{ such that } |\lambda_i| \leq 1 \text{ for all } i \right\}$$

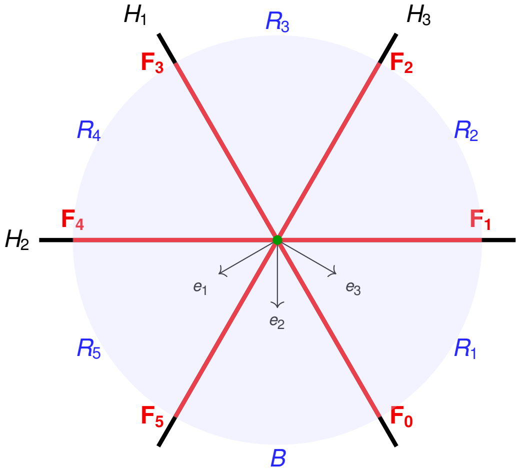
Theorem (Edelman '84, McMullen '71)

There is a bijection between $\mathscr{F}_{\mathcal{A}}$ and the nonempty faces of $Z_{\mathcal{A}}$ given by the map

$$\tau(F) = \left\{ v \in V \;\middle|\; v = \sum_{F(H_i)=0} \lambda_i e_i + \sum_{F(H_j)\neq 0} \mu_j e_j \right\}$$

$$\textit{where } |\lambda_i| \leq 1 \textit{ for all } i \textit{ and } \mu_j = F(H_j)$$

Zonotope - Construction example



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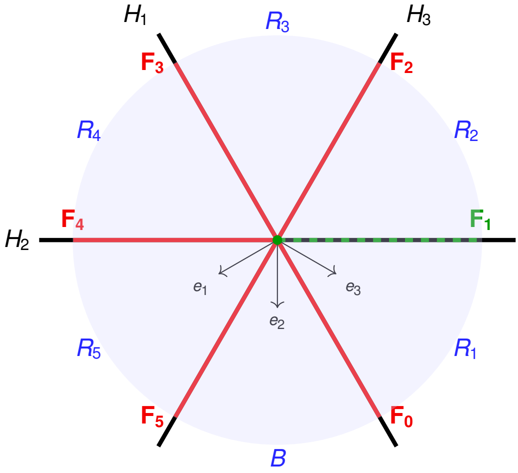

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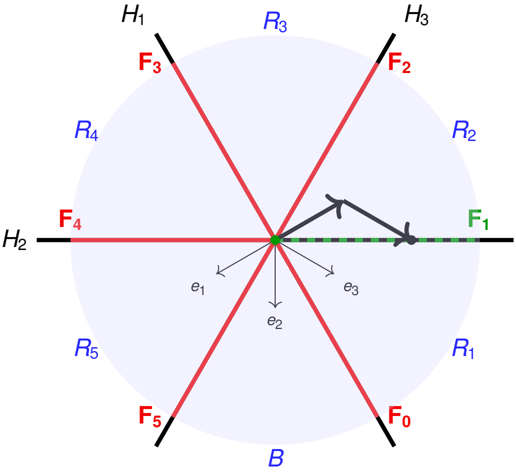

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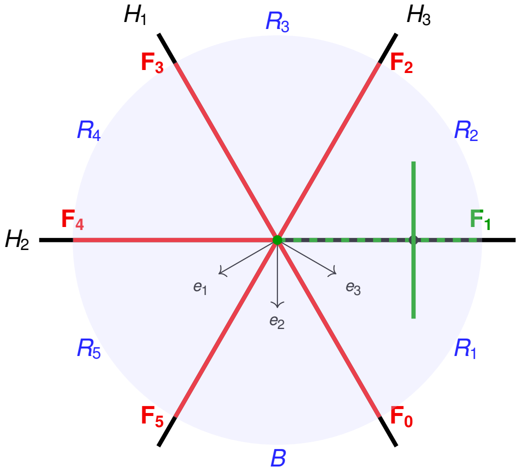

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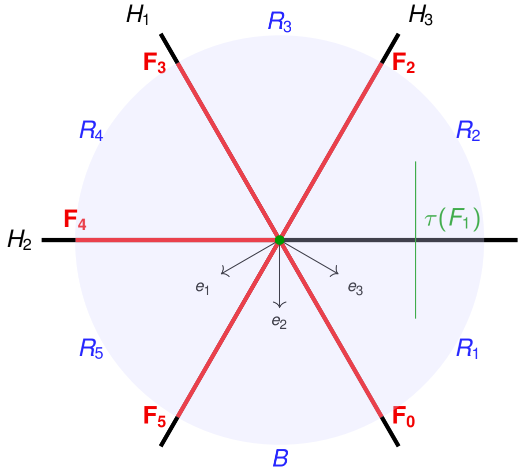

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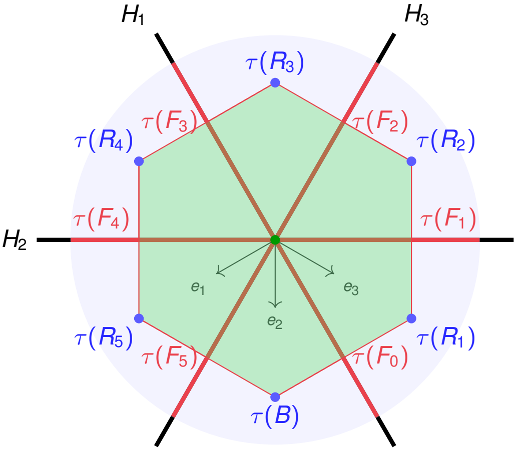
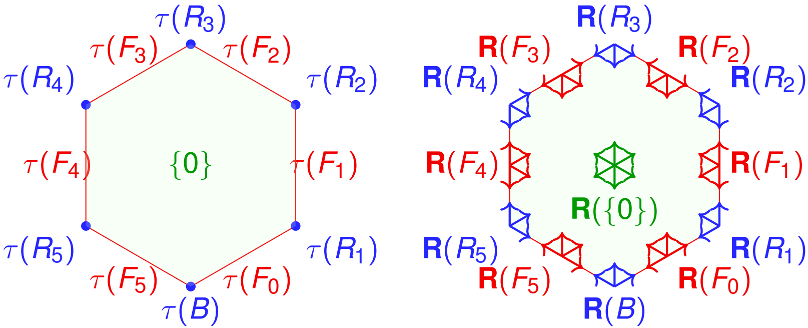


Root inversion sets

- *roots* $\Phi_{\mathcal{A}} := \{\pm e_1, \pm e_2, \ldots, \pm e_k\}$
- *root inversion set*
  $\mathbf{R}(F) := \{e \in \Phi_{\mathcal{A}} \mid \langle x, e \rangle \leq 0 \text{ for some } x \in F\}$.



Equivalent definitions

Theorem (D., Hohlweg, McConville, Pilaud '19+)

For $F, G \in \mathscr{F}_{\mathcal{A}}$ the following are equivalent:

- $m_F \leq_{\mathrm{PR}} m_G$ *and* $M_F \leq_{\mathrm{PR}} M_G$ *in poset of regions* $\mathrm{PR}(\mathcal{A}, B)$.
- *There exists a chain of covers in* $\mathrm{FW}(\mathcal{A}, B)$ *such that*

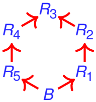
$$F = F_1 \lessdot F_2 \lessdot \cdots \lessdot F_n = G$$

- $F \leq_{\mathcal{L}} G$ *in terms of covectors* ($F(H) \geq G(H)\ \forall H \in \mathcal{A}$)
- $\mathbf{R}(F) \setminus \mathbf{R}(G) \subseteq \Phi_{\mathcal{A}}^{-}$ *and* $\mathbf{R}(G) \setminus \mathbf{R}(F) \subseteq \Phi_{\mathcal{A}}^{+}$.

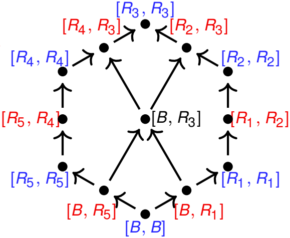
Warning!

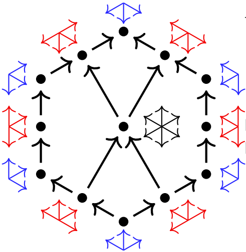
Next slide contains a lot of data. . . please procede with caution.

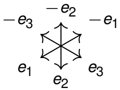
Equivalence for type A_2 Coxeter arrangement



$m_F \leq_{PR} m_G$
$M_F \leq_{PR} M_G$







$\mathbf{R}(F) \setminus \mathbf{R}(G) \subseteq \Phi^-_{\mathcal{A}}$
$\mathbf{R}(G) \setminus \mathbf{R}(F) \subseteq \Phi^+_{\mathcal{A}}$

$|\dim F - \dim G| = 1$ and
1. $F \subseteq G,\ M_F = M_G$, or
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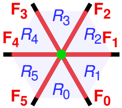
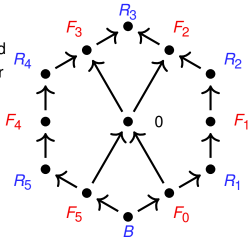
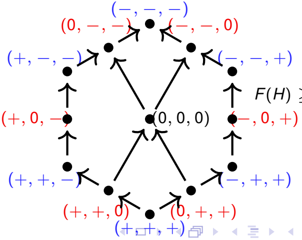
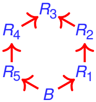






$F(H) \geq G(H)\ \forall H \in \mathcal{A}$

Equivalence for type A_2 Coxeter arrangement



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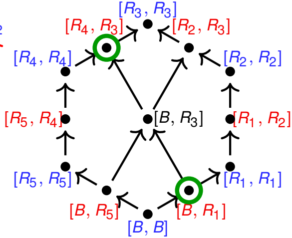

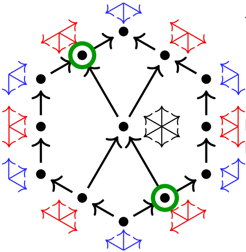

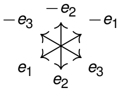

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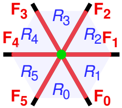

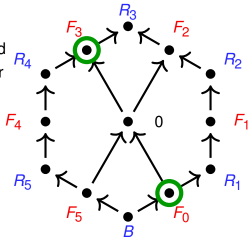

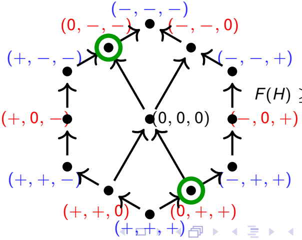

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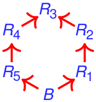
Equivalence for type A_2 Coxeter arrangement



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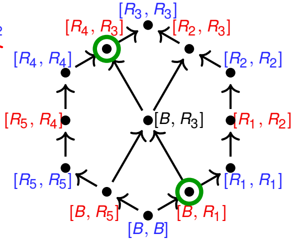

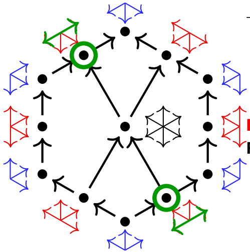

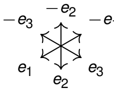

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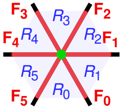

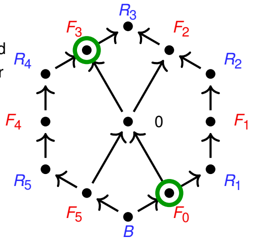

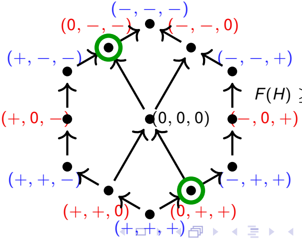

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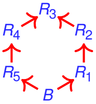
Equivalence for type A_2 Coxeter arrangement



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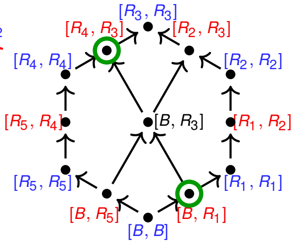

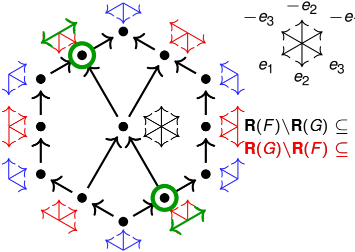

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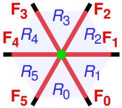

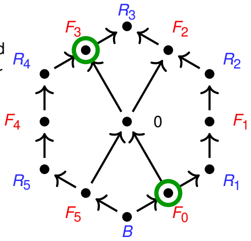

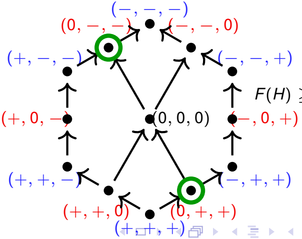

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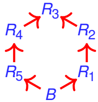

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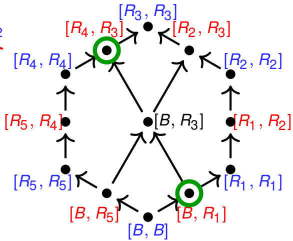

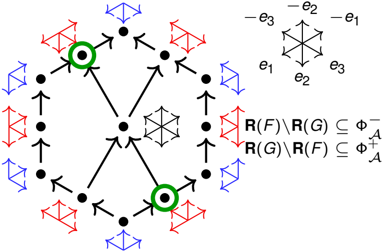

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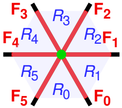

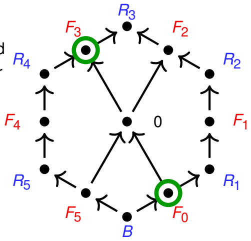

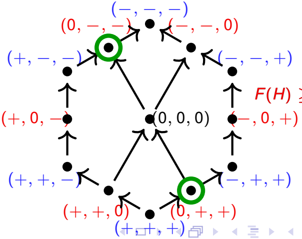

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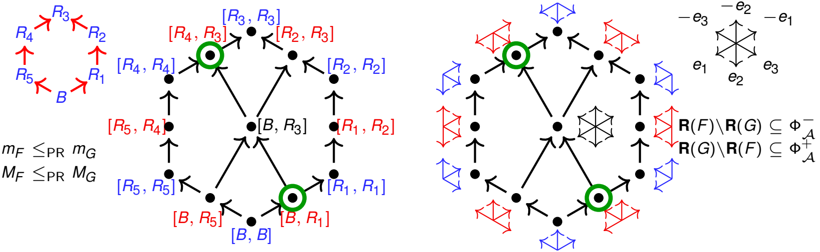

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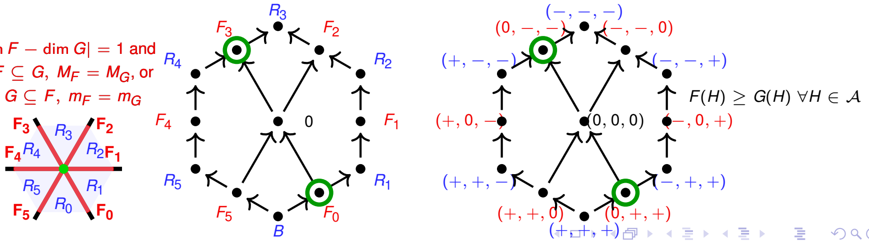

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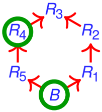

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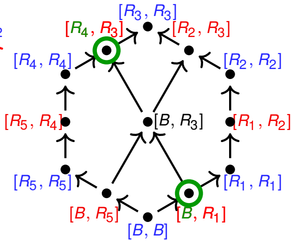

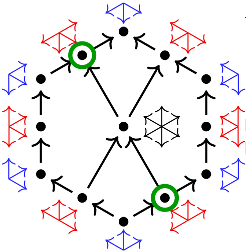

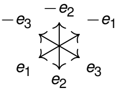

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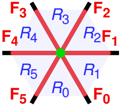

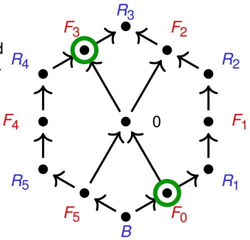

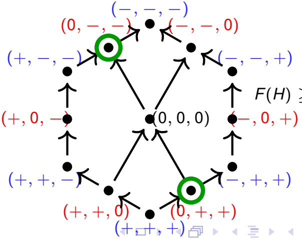

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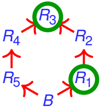

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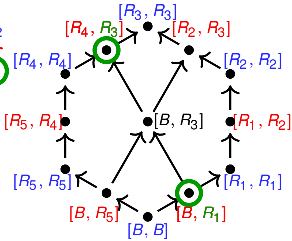

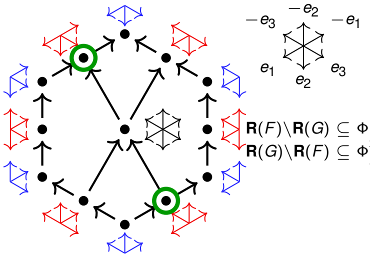

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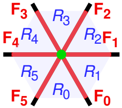
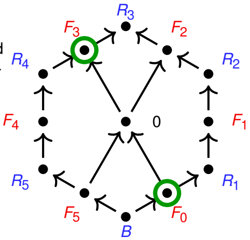
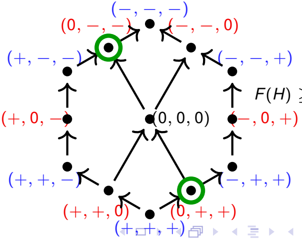






$F(H) \geq G(H)\ \forall H \in \mathcal{A}$

Facial weak order lattice

Theorem (D., Hohlweg, McConville, Pilaud '19+)

Let $\mathcal{A}$ be an arrangement and fix a base region B. If the poset of regions $\mathrm{PR}(\mathcal{A}, B)$ is a lattice then the facial weak order $\mathrm{FW}(\mathcal{A}, B)$ is a lattice.

Corollary (D., Hohlweg, McConville, Pilaud '19+)

The lattice of regions is a sublattice of the facial weak order lattice when $\mathcal{A}$ is simplicial.

Lattice proof - Joins

Proof uses two key components :

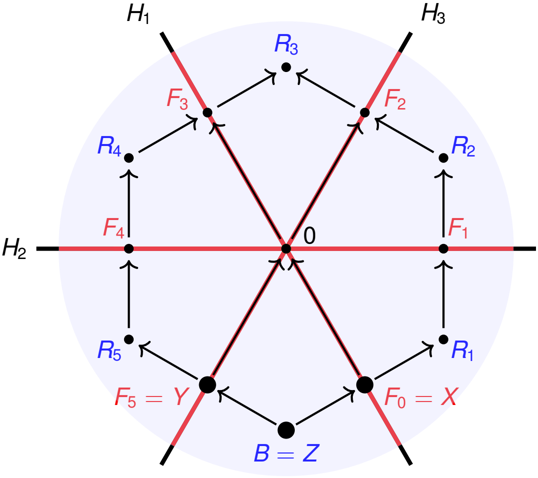
Lemma (Björner, Edelman, Ziegler '90)

1: If L is a finite, bounded poset such that $x \vee y$ exists whenever x and y both cover some $z \in L$, then L is a lattice.

2: Cover relation: $Z \lessdot X$ iff $|\dim X - \dim Z| = 1$ and $(X \subseteq Z,\ m_X = m_Z$ or $Z \subseteq X,\ M_Z = M_X)$. Then $Z \lessdot X$ and $Z \lessdot Y$ gives three cases:

1. $X \cup Y \subseteq Z$ and $\dim X = \dim Y = \dim Z - 1$,
2. $Z \subseteq X \cap Y$ and $\dim X = \dim Y = \dim Z + 1$, and
3. $X \subseteq Z \subseteq Y$ and $\dim X = \dim Z - 1 = \dim Y - 2$.

$X \cup Y \subseteq Z$ and $\dim X = \dim Y = \dim Z - 1$



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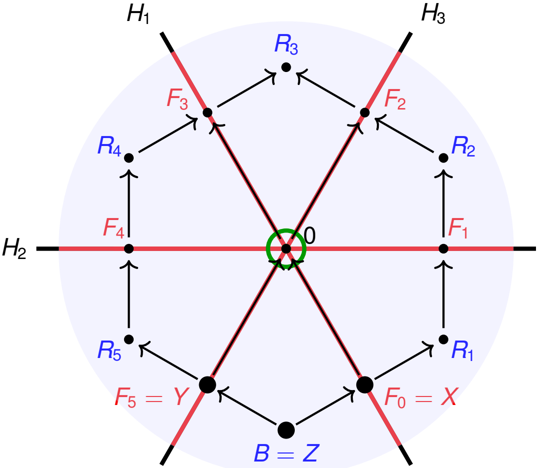

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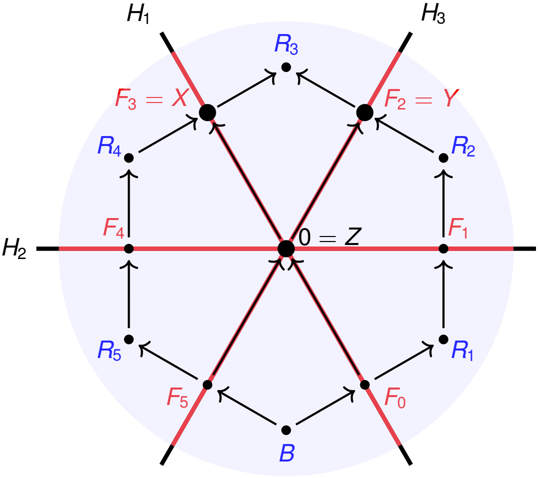

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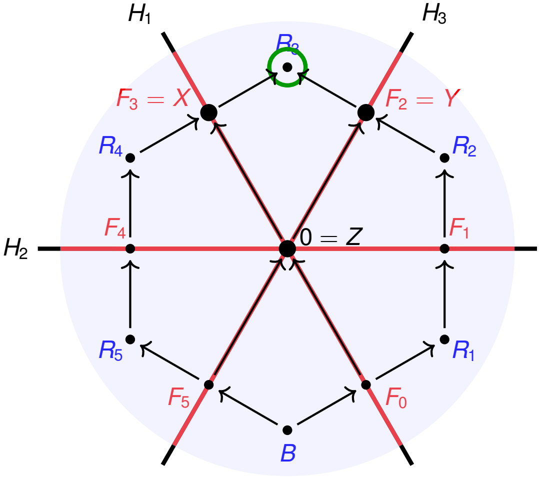

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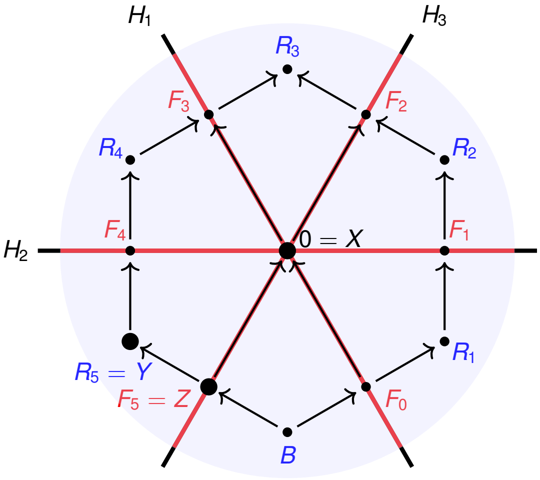

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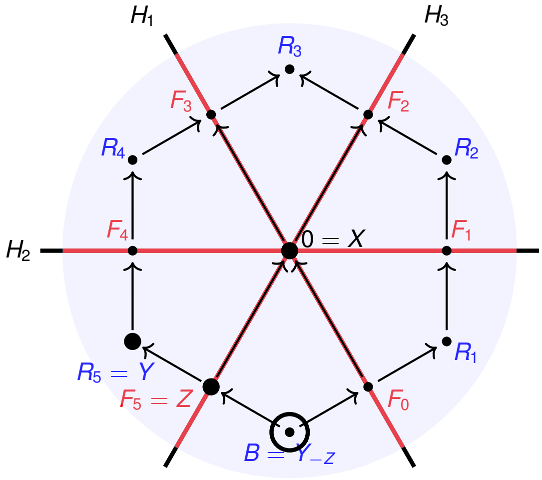

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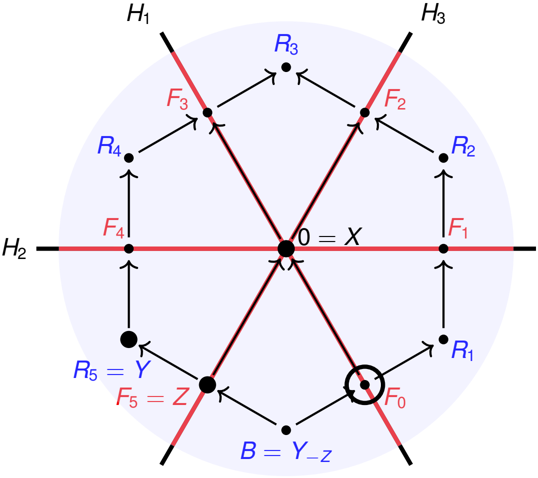

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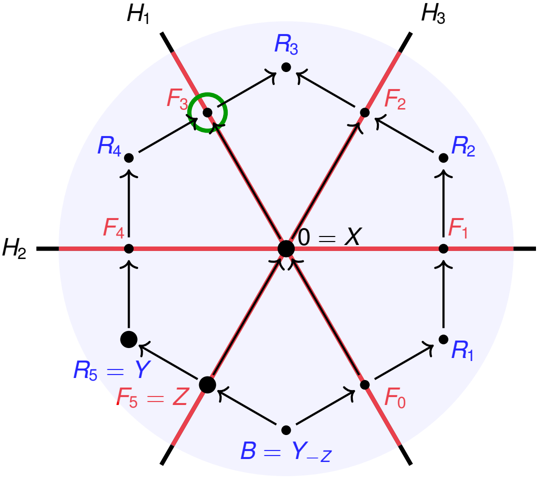
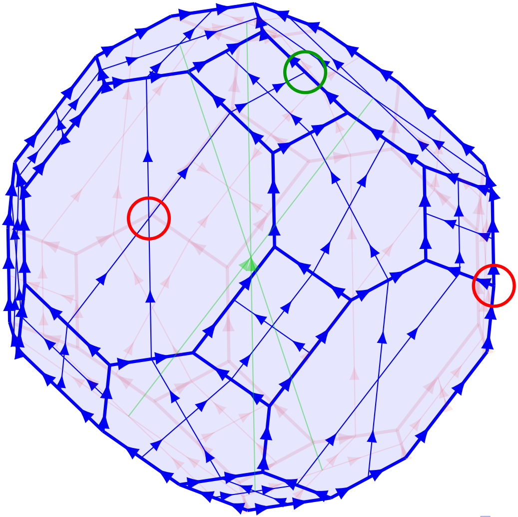


Example: B_3 Coxeter arrangement



Properties of the facial weak order

- The *dual* of a poset P is the poset P^{op} where $x \leq y$ in P iff $y \leq x$ in P^{op}. A poset is *self-dual* if $P \cong P^{op}$.
- A lattice is *semi-distributive* if $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$ and similarly for the meets.

Theorem (D., Hohlweg, McConville, Pilaud '19+)

The facial weak order $\mathrm{FW}(\mathcal{A}, B)$ *is self-dual. If furthermore,* $\mathcal{A}$ *is simplicial,* $\mathrm{FW}(\mathcal{A}, B)$ *is a semi-distributive lattice.*

Join-irreducible elements

- An element is *join-irreducible* if and only if it covers exactly one element.

Proposition (D., Hohlweg, McConville, Pilaud '19+)

If $\mathcal{A}$ *is simplicial and* F *a face with facial interval* $[m_F, M_F]$. *Then* F *is join-irreducible in* $\mathrm{FW}(\mathcal{A}, B)$ *if and only if* M_F *is join-irreducible in* $\mathrm{PR}(\mathcal{A}, B)$ *and* $\mathrm{codim}(F) \in \{0, 1\}$

Möbius function

Recall that the Möbius function is given by:

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y \\ -\sum_{x \leq z < y} \mu(x,z) & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}$$

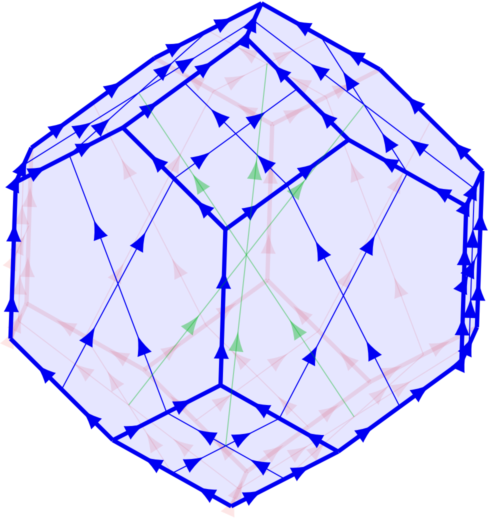
Proposition (D., Hohlweg, McConville, Pilaud '19+)

Let X and Y be faces such that $X \leq Y$ and let $Z = X \cap Y$.

$$\mu(X,Y) = \begin{cases} (-1)^{\mathrm{rk}(X)+\mathrm{rk}(Y)} & \text{if } X \leq Z \leq Y \text{ and } Z = X_{-Z} \cap Y \\ 0 & \text{otherwise} \end{cases}$$

Further Works

- Can we explicitly state the join/meet of two elements?
- Composition gives us a left regular band structure. Can we generalize everything to left regular bands?



Thank you!