Enumerating Weyl Cones of Shi Arrangements

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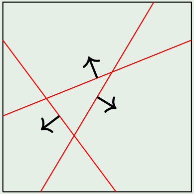
Hyperplane Arrangements

- $(V, \langle \cdot, \cdot \rangle)$ - n-dim real Euclidean vector space.
- A *hyperplane* H is a codim 1 subspace of V.
- A *(hyperplane) arrangement* is a *finite* collection of hyperplanes.

Example

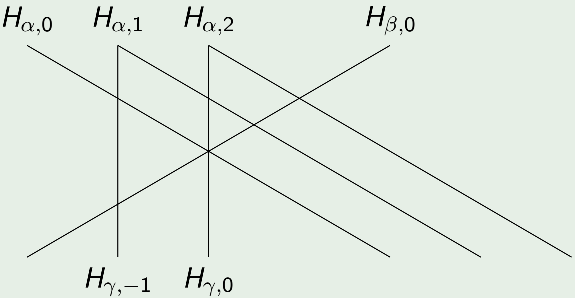


Hyperplanes and vectors

For $\alpha \in \mathbb{R}^n$ a vector.

- $H_{\alpha,k} = \{v \in \mathbb{R}^n \mid \langle \alpha, v \rangle = k\}$ - hyperplane.
- $H_\alpha = H_{\alpha,0}$ - central hyperplane.
- s_α - reflection fixing H_α pointwise.

Example



Root Systems

Definition

A *root system* Φ is (finite) collection of nonzero vectors satisfying:

1. $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for every $\alpha \in \Phi$.
2. $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.
3. $\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

The $\alpha \in \Phi$ are called *roots*.

- Φ^+ - Positive roots
- Φ^- - Negative roots
- Δ - Simple roots
- $W = \langle S \rangle$, $S = \{s_\alpha \mid \alpha \in \Delta\}$ - Weyl group.

Coxeter and Shi Arrangements

Definitions

A *Coxeter arrangement* is the arrangement for a root system Φ:

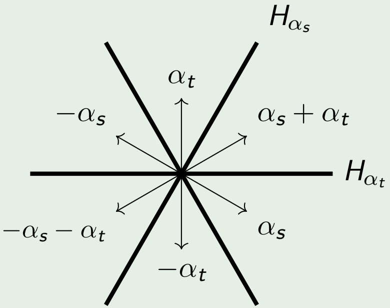
$$\mathcal{A}(\Phi) = \{ H_\alpha \mid \alpha \in \Phi^+ \}.$$

A *Shi arrangement* is the Coxeter arrangement together with a positive unit translate of each hyperplane:

$$\mathrm{Shi}(\Phi) = \{ H_{\alpha,k} \mid \alpha \in \Phi^+, k \in \{0,1\} \}$$

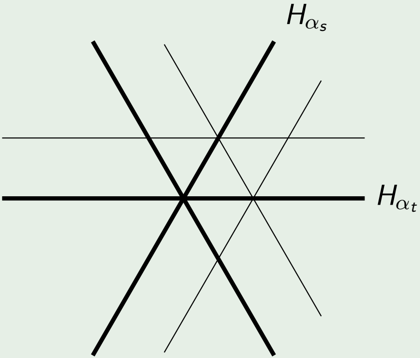
A_2 example

Example (Coxeter Arrangement)



A_2 example

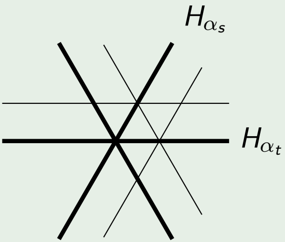
Example (Shi Arrangement)



Regions

A *region* is a (open) connected component of the vector space with the hyperplanes removed.

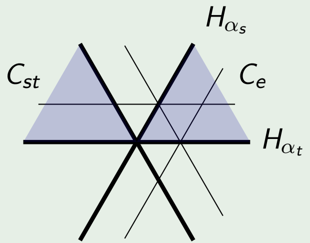
Example (Shi Arrangement)



Weyl cone

A *cone* is an intersection of (open) half-spaces of (some) hyperplanes.
For Shi(Φ), the regions of the Coxeter subarrangement are in bijection with the elements of W. These regions define cones called *Weyl cones*. The cone associated to the identity is commonly referred to as the *dominant cone*.

Example (Shi Arrangement)



Question:

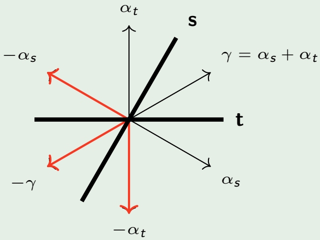
How many regions are in each Weyl cone?

Inversion Sets

The *(left) inversion sets* is the set

$$N(w) = \Phi^+ \cap w(\Phi^-).$$

Example



$$\begin{aligned} N(ts) &= \Phi^+ \cap ts(\Phi^-) \\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\} \\ &= \{\alpha_t, \gamma\} \end{aligned}$$

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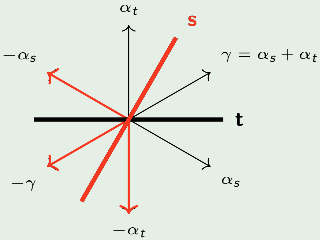

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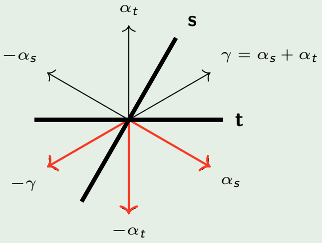

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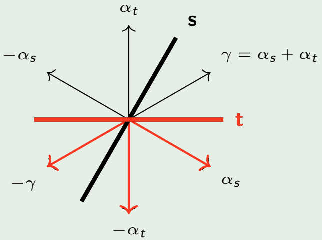

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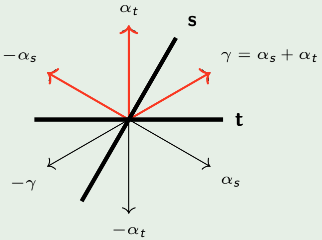

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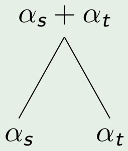
Root Poset

Definition

The *root poset* $(\Phi^+, \leq)$ is the poset where

$$\alpha < \beta \iff \beta - \alpha \in \mathbb{N}\Delta$$

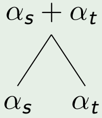
Example



Antichain

An *antichain* in a poset is a set of pairwise incomparible elements.

Example



There are 5 antichains:

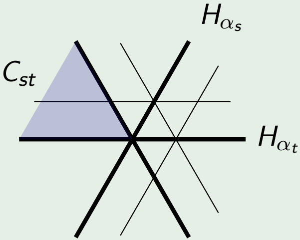
$$\varnothing, \{\alpha_s\}, \{\alpha_t\}, \{\alpha_s + \alpha_t\}, \{\alpha_s, \alpha_t\}$$

Number of regions using antichains

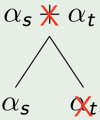
Theorem (Armstrong, Reiner, Rhoades 2015)

The number of regions in a Weyl cone C_w is equal to the number of antichains in the subposet of the root poset $(\Phi^+, \leq)$ restricted to $\Phi^+ \setminus N(w^{-1})$.

Example (A_2 Shi Arrangement)



$N(ts) = \{\alpha_t, \alpha_s + \alpha_t\}$



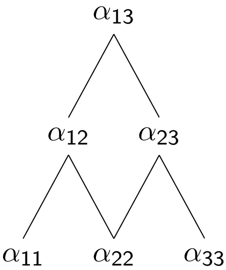
2 antichains: $\varnothing$, $\{\alpha_s\}$

Diagrams (type A)

Shorthand: $\alpha_{ij} = \sum_{k=i}^{j} \alpha_k$

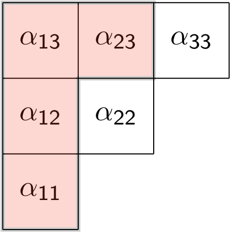
α_{13}	α_{23}	α_{33}
α_{12}	α_{22}	
α_{11}		

$\leftrightarrow$



Subdiagrams

A *subdiagram* is a set B of boxes such that if $b \in B$ then every box above and to the left are also in B.



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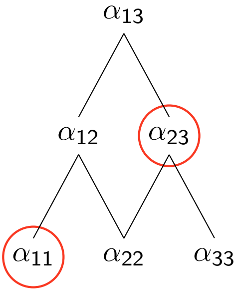


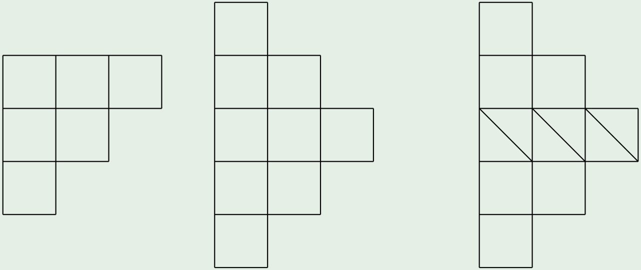
If a box is in the bottom right corner of the subdiagram, it is in antichain.

Subdiagrams

Theorem (Shi 1995)

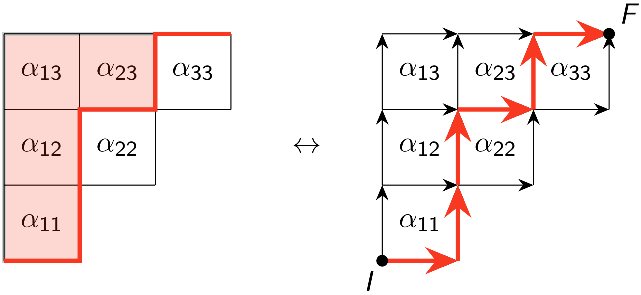
Let Λ be the diagram associated to a Coxeter group W with root system Φ. Then there is a bijection between number of subdiagrams of Λ and antichains in $(\Phi^+, \leq)$.

Example (A_3, B_3, D_4)



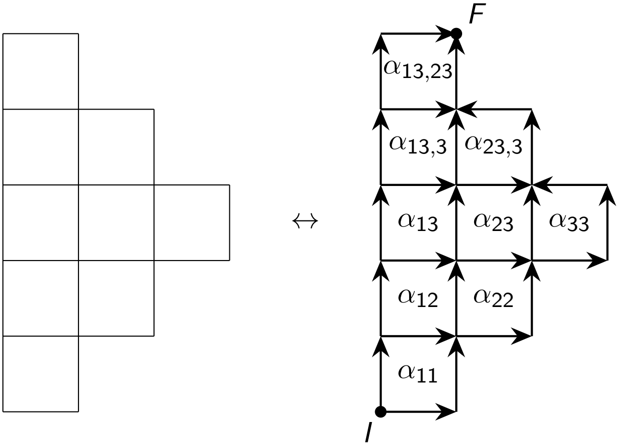
Diagrams to Digraphs - Type A

Shorthand: $\alpha_{ij} = \sum_{k=i}^{j} \alpha_k$

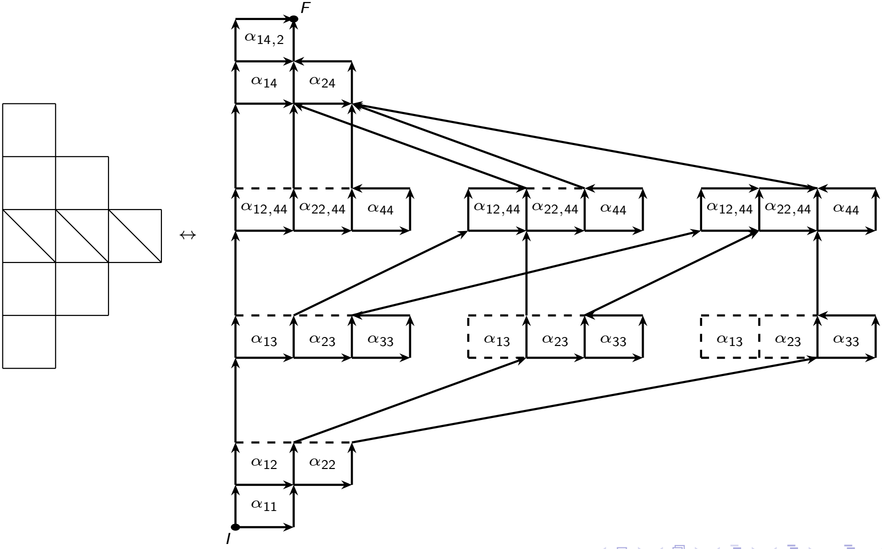


Diagrams to Digraphs - Type B

Shorthand: $\alpha_{ij,k\ell} = \alpha_{ij} + \alpha_{k\ell}$



Diagrams to Digraphs - Type *D*

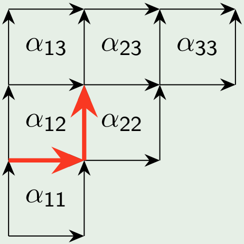


Corners

For each $\alpha \in \Phi^+$ we let Π_α be the set of subpaths of Γ which go under and to the right of α.

Example

$\Pi_{\alpha_{12}}$ is associated to the following subpath.



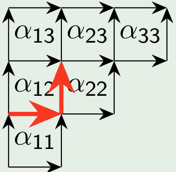
Digraph solution

$$\text{Let} \qquad \Pi_w = \bigcup_{\alpha \in N(w^{-1})} \Pi_\alpha$$

Theorem (D., Tzanaki 2023)

Let Γ be the digraph associated to W with root system Φ. There is a bijection between paths in Γ which don't contain subpaths in Π_w and antichains in the root poset $(\Phi^+, \leq)$ restricted to $\Phi^+ \setminus N(w^{-1})$.

Example



But..

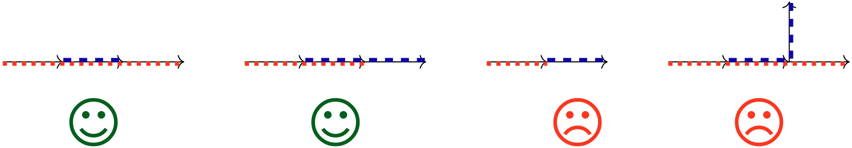
How does this help?

Overlapping paths

Γ a directed graph. $\pi = (v_1, e_1, v_2, e_2, \ldots, e_{n-1}, v_n)$ be a path. Let $I_\pi = v_1$ and $F_\pi = v_n$. Γ is *acyclic* if there are no paths such that $I_\pi = F_\pi$.

Two paths π and $\pi' = (u_1, f_1, \ldots, f_{m-1}, u_m)$ *overlap* if:

- π is a subpath of π', or
- there exists some $i \in [n-1]$ such that for all $j \in [n-i]$, then $e_{i+j-1} = f_j$ (the final i *edges* in π coincide with the first i edges of π').



Number of paths

A collection of paths Π is *non-overlapping* if there does not exist any $\pi, \pi' \in \Pi$ such that π overlaps π'.
Let $\gamma(v \to v')$ be the number of paths from v to v'.

Theorem (D., Tzanaki 2023)

Let I and F be two arbitrary vertices in an acyclic digraph Γ. Let Π be a collection of non-overlapping paths. Then the number of paths from I to F which do not contain a path in Π as a subpath is equal to:

$$\det \begin{pmatrix} 1 & \gamma(F_2 \to I_1) & \cdots & \gamma(F_n \to I_1) & \gamma(I \to I_1) \\ \gamma(F_1 \to I_2) & 1 & \cdots & \gamma(F_n \to I_2) & \gamma(I \to I_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma(F_1 \to I_n) & \gamma(F_2 \to I_n) & \cdots & 1 & \gamma(I \to I_n) \\ \gamma(F_1 \to F) & \gamma(F_2 \to F) & \cdots & \gamma(F_n \to F) & \gamma(I \to F) \end{pmatrix}$$

Path enumeration

Theorem (André 1887)

Let Γ be the infinite digraph of $\mathbb{Z}^2$ with vertical edges pointing north and horizontal edges pointing east. Label every vertex of Γ by its respective coordinates in $\mathbb{Z}^2$. Then the number of paths from (x_1, y_1) to (x_2, y_2) weakly above the $x = y$ diagonal is given by:
If $x_1 \leq x_2$ and $y_1 \leq y_2$:

$$\binom{x_2 + y_2 - x_1 - y_1}{y_2 - y_1} - \binom{x_2 + y_2 - x_1 - y_1}{y_2 - x_1 + 1}$$

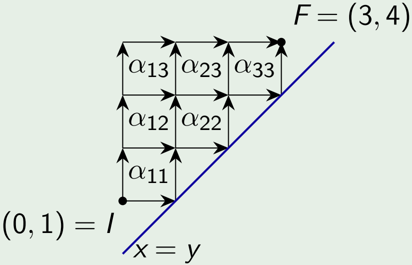
and 0 otherwise.

Type A

Let Γ be the infinite digraph of $\mathbb{Z}^2$.

- $I = (0, 1)$ and $F = (n, n + 1)$.
- $\alpha_{ij} = \sum_{k=i}^{j} \alpha_k \in \Phi, \Rightarrow \pi_{ij} : (i - 1, j) \to (i, j) \to (i, j + 1)$.

Example



A_5 example

Let W be the A_5 Coxeter arrangement and $w = s_5 s_2 s_4 s_3 s_1$. Then

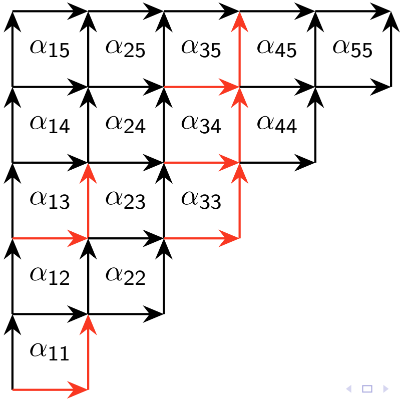
$$N(w^{-1}) = \{\alpha_{11}, \alpha_{33}, \alpha_{34}, \alpha_{13}, \alpha_{35}\}$$

$$\begin{aligned}
\alpha_{11} &\leftrightarrow (0,1) \to (1,1) \to (1,2) \\
\alpha_{33} &\leftrightarrow (2,3) \to (3,3) \to (3,4) \\
\alpha_{34} = \alpha_3 + \alpha_4 &\leftrightarrow (2,4) \to (3,4) \to (3,5) \\
\alpha_{13} = \alpha_1 + \alpha_2 + \alpha_3 &\leftrightarrow (0,3) \to (1,3) \to (1,4) \\
\alpha_{35} = \alpha_3 + \alpha_4 + \alpha_5 &\leftrightarrow (2,5) \to (3,5) \to (3,6)
\end{aligned}$$

A_5 example cont.

$$N(w^{-1}) = \{\alpha_{11}, \alpha_{33}, \alpha_{34}, \alpha_{13}, \alpha_{35}\}$$

$$\Pi_w = \left\{\pi_{ij} \mid \alpha_{ij} \in N(w^{-1})\right\}$$



A_5 example cont.

The number of regions in C_w is equal to

$$\det \begin{pmatrix} 1 & \gamma((3,4)\to(0,1)) & \gamma((3,5)\to(0,1)) & \gamma((1,4)\to(0,1)) & \gamma((3,6)\to(0,1)) & \gamma((0,1)\to(0,1)) \\ \gamma((1,2)\to(2,3)) & 1 & \gamma((3,5)\to(2,3)) & \gamma((1,4)\to(2,3)) & \gamma((3,6)\to(2,3)) & \gamma((0,1)\to(2,3)) \\ \gamma((1,2)\to(2,4)) & \gamma((3,4)\to(2,4)) & 1 & \gamma((1,4)\to(2,4)) & \gamma((3,6)\to(2,4)) & \gamma((0,1)\to(2,4)) \\ \gamma((1,2)\to(0,3)) & \gamma((3,4)\to(0,3)) & \gamma((3,5)\to(0,3)) & 1 & \gamma((3,6)\to(0,3)) & \gamma((0,1)\to(0,3)) \\ \gamma((1,2)\to(2,5)) & \gamma((3,4)\to(2,5)) & \gamma((3,5)\to(2,5)) & \gamma((1,4)\to(2,5)) & 1 & \gamma((0,0)\to(2,5)) \\ \gamma((1,2)\to(5,6)) & \gamma((3,4)\to(5,6)) & \gamma((3,5)\to(5,6)) & \gamma((1,4)\to(5,6)) & \gamma((3,6)\to(5,6)) & \gamma((0,1)\to(5,6)) \end{pmatrix}$$

A_5 example cont.

The number of regions in C_w is equal to

$$\det\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \binom{0}{0}-\binom{0}{2} \\ \binom{2}{1}-\binom{2}{3} & 1 & 0 & 0 & 0 & \binom{4}{2}-\binom{4}{4} \\ \binom{3}{2}-\binom{3}{4} & 0 & 1 & \binom{1}{0}-\binom{1}{4} & 0 & \binom{5}{3}-\binom{5}{5} \\ 0 & 0 & 0 & 1 & 0 & \binom{2}{2}-\binom{2}{4} \\ \binom{4}{3}-\binom{4}{5} & 0 & 0 & \binom{2}{1}-\binom{2}{5} & 1 & \binom{6}{4}-\binom{6}{6} \\ \binom{8}{4}-\binom{8}{6} & \binom{4}{2}-\binom{4}{4} & \binom{3}{1}-\binom{3}{4} & \binom{6}{2}-\binom{6}{6} & \binom{2}{0}-\binom{2}{4} & \binom{9}{4}-\binom{9}{7} \end{pmatrix}$$

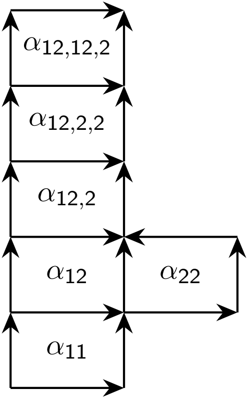
A_5 example cont.

The number of regions in C_w is equal to

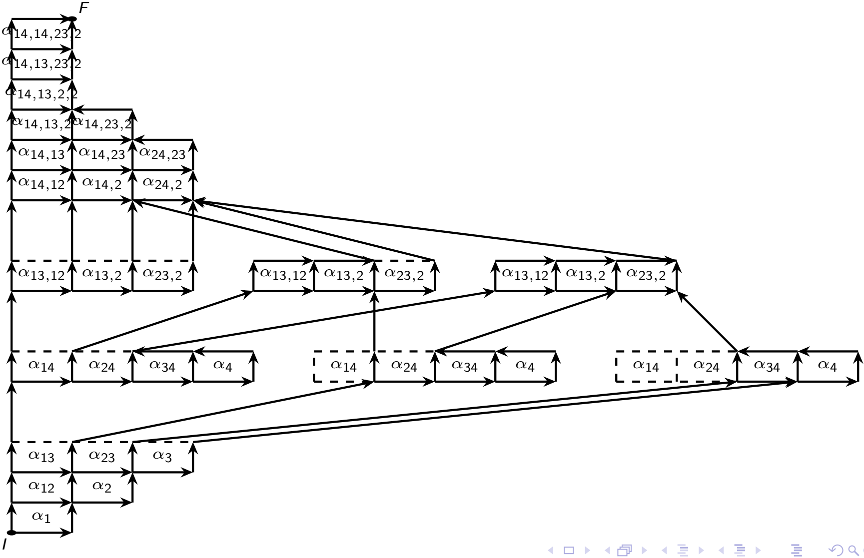
$$\det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 5 \\ 3 & 0 & 1 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 4 & 0 & 0 & 2 & 1 & 14 \\ 42 & 5 & 3 & 14 & 1 & 132 \end{pmatrix} = 38$$

THANK YOU

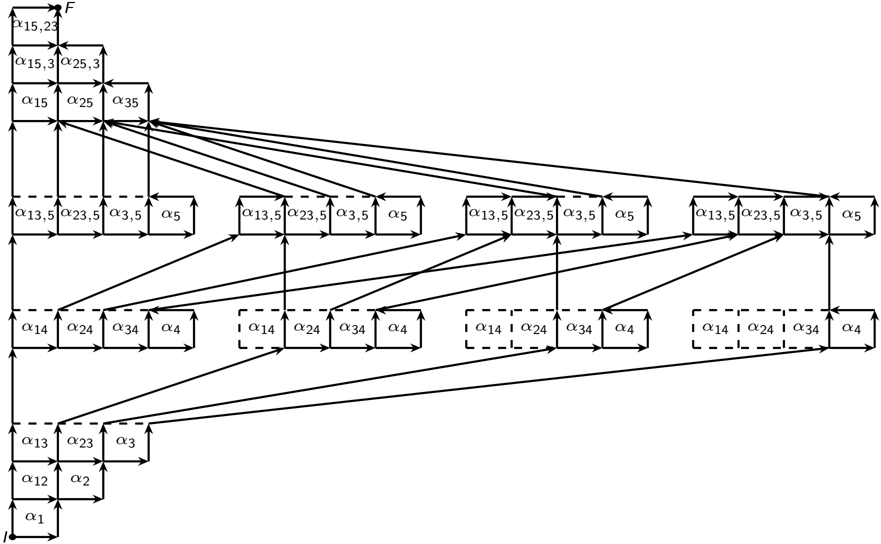
Digraph - Type G_2



Digraph - Type F_4



Digraphs - Type D_5



Digraphs - Type E_6