Bruhat Preclosure

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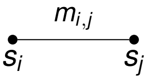
Coxeter groups

- *Coxeter System* (W, S)

$$W := \langle S \mid (s_i s_j)^{m_{i,j}} = e \text{ for } s_i, s_j \in S \rangle$$

$m_{i,j} \in \mathbb{N}^\star \cup \{\infty\}$ and $m_{i,j} = 1$ only if $i = j$.

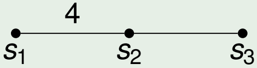
- *Coxeter diagram* Γ_W: vertices S and edges labelled $m_{i,j}$ when $m_{i,j} > 3$.



Example

$$W_{B_3} = \left\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \right\rangle$$

Γ_{B_3} :



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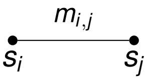
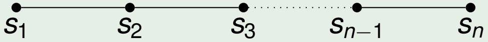


Example

$W_{A_n} = S_{n+1}$, symmetric group.

Γ_{A_n} :



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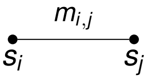
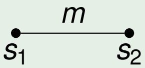


Example

$W_{I_2(m)} = \mathcal{D}(m)$, dihedral group of order $2m$.

$\Gamma_{I_2(m)}$:



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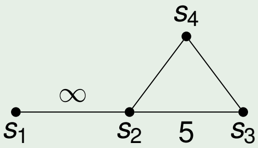
Example

W, an infinite Coxeter group.

Γ :



Words

(W, S) a Coxeter system.

- $w = s_1 \dots s_n \in W$ for $s_i \in S$.
- w has *length* n, $\ell(w) = n$, if n is minimal.
- If $\ell(w) = n$ then $w = s_1 \dots s_n$ is a *reduced expression*.

Example

Γ_{A_2} :

Then $\ell(stst) = 2$

$$stst = tstt = ts$$

since

$$(st)^3 = e \Rightarrow sts = tst.$$

ts is a reduced expression of $stst$.

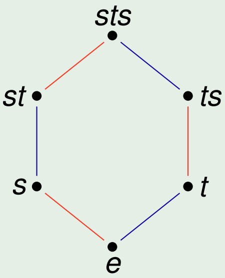
Weak order

- *(right) weak order* is the prefix order on elements:

$$u \leq_R v \iff \ell(v) = \ell(u) + \ell(u^{-1}v)$$

Example

Γ_{A_2} :

Inversion sets

$T = \cup wSw^{-1}$ is the set of *reflections*.

For $w \in W$, *(left) inversion set*:

$$N(w) = \{t \in T \mid \ell(tw) < \ell(w)\}$$

Example

Γ_{A_2} : 

Recall $\ell(ts) = 2$. Then

$$N(ts) = \{t, tst\}$$

since

$$(t)(ts) = tts = s$$
$$(tst)(ts) = tstts = t$$

Generating inversion sets

For $w = s_1 \dots s_n$ a reduced expression:

$$N(w) = \{s_1, s_1 s_2 s_1, s_1 s_2 s_3 s_2 s_1, \cdots, s_1 s_2 \dots s_n \dots s_2 s_1\}$$

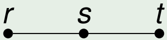
Example

Γ_{A_2} :

$$N(ts) = \{t, tst\}$$

Example

Γ_{A_3} :



$$N(rstsr) = \{r, rsr, rstsr, rststsr = t, rstsrstsr = tst\}$$

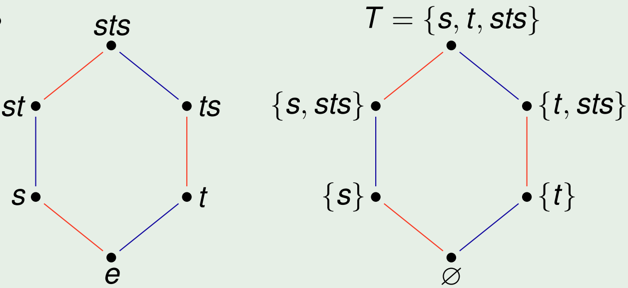
Weak order - Round 2

- The *(right) weak order* is the prefix order on elements:

$$u \leq_R v \iff \ell(v) = \ell(u) + \ell(u^{-1}v) \iff \mathbf{N(u)} \subseteq \mathbf{N(v)}$$

Example

Γ_{A_2} :



(Semi)lattices

(W, S) a Coxeter system and $u, v \in W$.

- *Meet* (greatest lower bound) - $u \wedge_R v$
- *Join* (least upper bound) - $u \vee_R v$
- *Meet-semilattice* every two elements has a meet.
- *Join-semilattice* every two elements has a join.
- *Lattice* every two elements has a meet and join.

Theorem (Björner [1984])

The weak order is a meet-lattice. If W is finite, it's a lattice.

Lattice

- *Meet* (greatest lower bound) - $u \wedge_R v$
- *Join* (least upper bound) - $u \vee_R v$
- *Meet-semilattice* every two elements has a meet.
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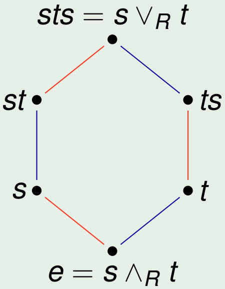
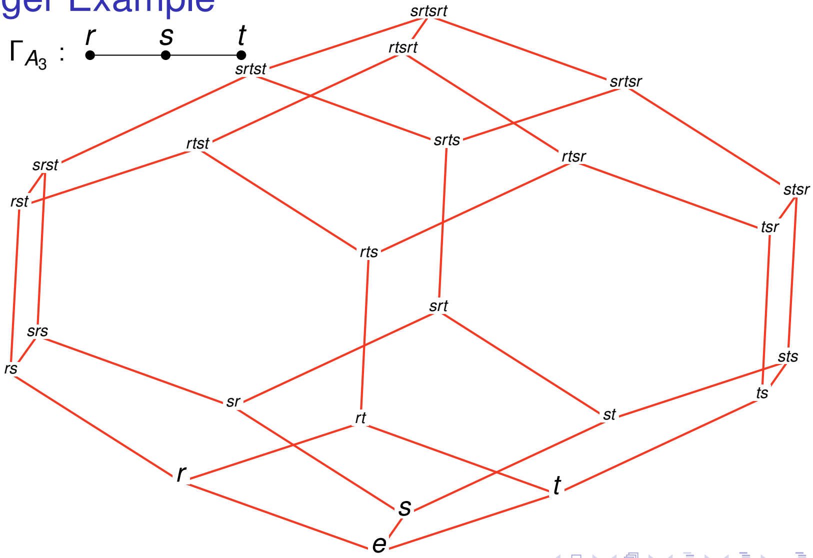
Example

Γ_{A_2} :

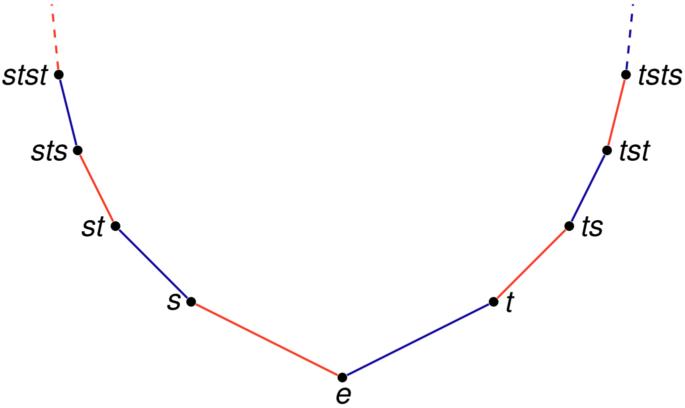


Larger Example



Infinite example

$\Gamma_{\tilde{A}_1}$:



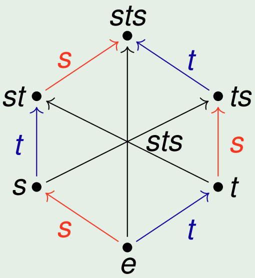
Bruhat graph Ω

- Vertices: Elements of W
- Edges: (u, v) if
  - $v = ut$ for some $t \in T$
  - $\ell(u) \leq \ell(v)$.

Example

Γ_{A_2} : 

.



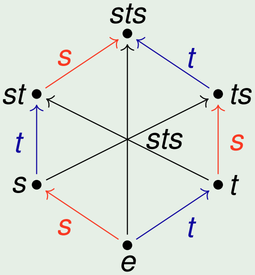
A-Path

For $A \subseteq T$, *A-path* a directed path in Ω following only edges labelled by elements in A.

Example

Γ_{A_2} :

$A = \{s, sts\} \subseteq \{s, sts, t\} = T$



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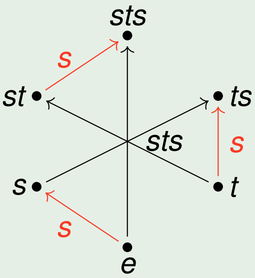

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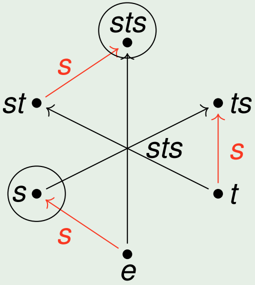

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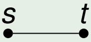
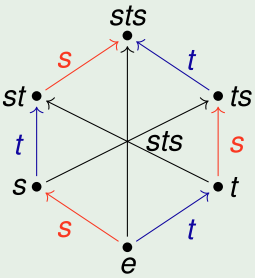
Γ_{A_2} :  $A = \{sts\} \subseteq \{s, sts, t\} = T$



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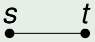
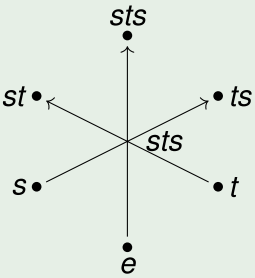
Γ_{A_2} :  $A = \{sts\} \subseteq \{s, sts, t\} = T$



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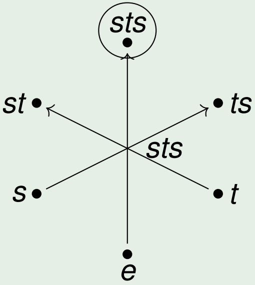

Bruhat preclosure

For $A \subseteq T$

$$\overline{A} = \{t \in T \mid \text{there exists an } A\text{-path from } e \text{ to } t\}$$

Example

Γ_{A_2} :

$$\overline{\{s, sts\}} = \{s, sts\}$$
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$$\overline{\{s, t\}} = \{s, sts, t\} = T$$

Bruhat preclosure

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Example

Γ_{A_2} :

$$\overline{\{s, sts\}} = \{s, sts\} = N(st)$$

$$\overline{\{sts\}} = \{sts\}$$

$$\overline{\{s, t\}} = \{s, sts, t\} = T = N(sts) = N(s \vee_R t)$$

Dyer's conjecture

For $A \subseteq T$ and $u, v \in W$:

1. $\overline{A}$ is a closure operator.
2. $\overline{N(u) \cup N(v)} = N(u \vee_R v)$.

Dyer's conjecture

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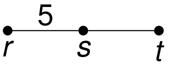
Counterexample

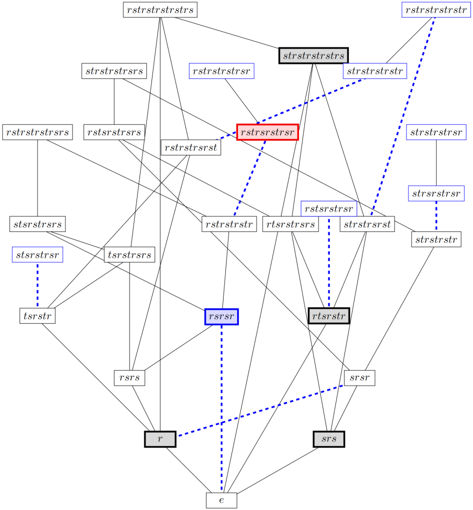
Γ_{H_3} :



$$A = \{r,\ srs,\ tsrsrst,\ strstrstrstrs\}$$

$$\overline{A} = A \cup \{srsrs\}$$

$$\overline{\overline{A}} = \overline{A} \cup \{rstrsrstrsr\}$$



Dyer's conjecture

For $A \subseteq T$ and $u, v \in W$:

1. ~~$\overline{A}$ is a closure operator.~~ Can we do something about this?
2. $\overline{N(u) \cup N(v)} = N(u \vee_R v)$.

Reflection order

A *reflection order* $(T, \preccurlyeq)$ such that for every $r, s \in T$, $W' = \langle r, s \rangle$:

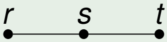
$$r \prec rsr \prec \cdots \prec srs \prec s$$

or

$$s \prec srs \prec \cdots \prec rsr \prec r$$

Example

Γ_{A_3} :



$$r \preccurlyeq t \preccurlyeq rstsr \preccurlyeq sts \preccurlyeq rsr \preccurlyeq s$$

$$r \preccurlyeq rsr \preccurlyeq rstsr \preccurlyeq s \preccurlyeq sts \preccurlyeq t$$

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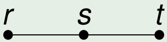

$$\langle s, t \rangle \Rightarrow \{s, t, sts\}$$

$$r \preccurlyeq t \preccurlyeq rstsr \preccurlyeq sts \preccurlyeq rsr \preccurlyeq s$$

$$r \preccurlyeq rsr \preccurlyeq rstsr \preccurlyeq s \preccurlyeq sts \preccurlyeq t$$

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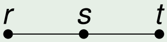

$$\langle r, s \rangle \Rightarrow \{r, s, rsr\}$$

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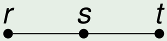

$$\langle r, sts \rangle \Rightarrow \{r, sts, rstsr\}$$

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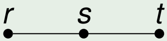


$$\langle rsr, t \rangle \Rightarrow \{rsr, t, rstsr\}$$

$$r \preccurlyeq t \preccurlyeq rstsr \preccurlyeq sts \preccurlyeq rsr \preccurlyeq s$$

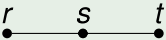
$$r \preccurlyeq rsr \preccurlyeq rstsr \preccurlyeq s \preccurlyeq sts \preccurlyeq t$$

Initial section

Subset of T containing an "initial part" of *some* reflection order.
Finite = Inversion sets

Example

Γ_{A_3} :



$$r \preccurlyeq t \preccurlyeq rstsr \preccurlyeq sts \preccurlyeq rsr \preccurlyeq s$$

$$r \preccurlyeq rsr \preccurlyeq rstsr \preccurlyeq s \preccurlyeq sts \preccurlyeq t$$

$\{r, t\}$	$\{r, rsr, rstsr\}$	$\{r, t, rstsr, sts\}$	$\varnothing$	T
$N(rt)$	$N(rst)$	$N(rtsr)$	$N(e)$	$N(rstrst)$

Non-finite initial section: $\tilde{A}_1$

Subset of T containing an "initial part" of *some* reflection order.

Example

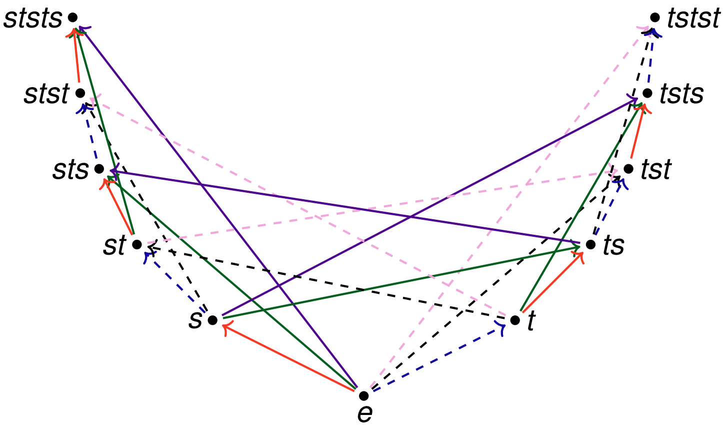
$\Gamma_{\tilde{A}_1}$:

$$s \prec sts \prec ststs \prec \cdots \prec tstst \prec tst \prec t$$

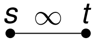
$$\text{Initial section: } \{w \in T \mid w \text{ begins with } s\}$$

$\tilde{A}_1$ $\Gamma_{\tilde{A}_1}$: $s \overset{\infty}{\text{—}} t$

s : red t : blue sts : green tst : black $ststs$: purple $tstst$: pink



$\tilde{A}_1$ $\Gamma_{\tilde{A}_1}$:



$N(st) = \{s, sts\}$

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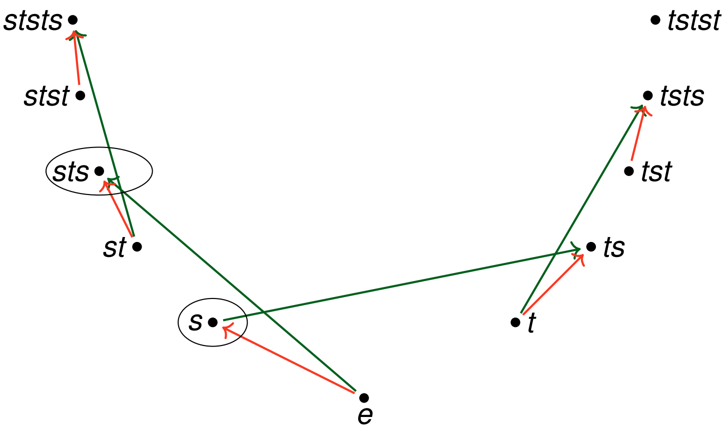
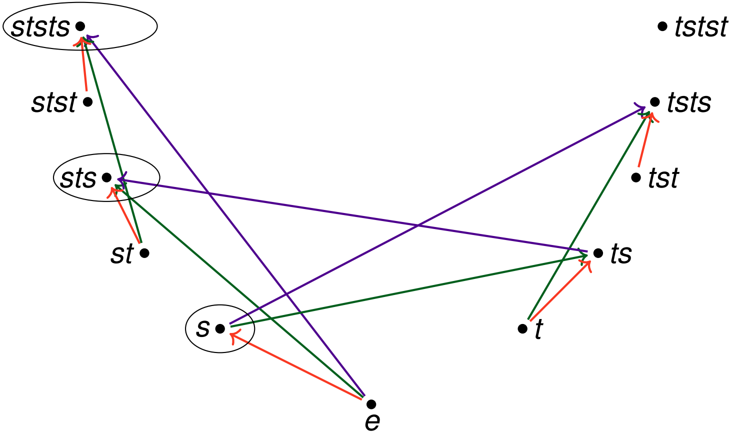


$\tilde{A}_1$ $\Gamma_{\tilde{A}_1}$: $s \overset{\infty}{—} t$ $\{w \in T \mid w \text{ begins with } s\}$

s : red t : blue sts : green tst : black $ststs$: purple $tstst$: pink

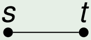


Initial sections

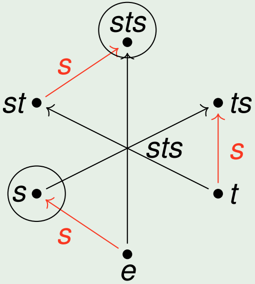
Theorem (D. 2025+)

If $A \subseteq T$ is an initial section, then $\overline{A} = A$.

Example

Γ_{A_2} : 

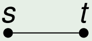
$A = N(st) = \{s, sts\} \subseteq \{s, sts, t\} = T$.



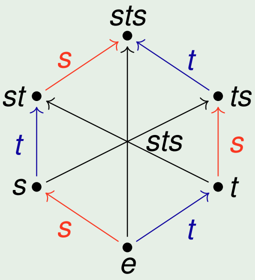
Twisted Bruhat

For $A \subseteq T$, flip the edges labelled with elements of A.

Example

Γ_{A_2} : 

$N(st) = \{s, sts\} \subseteq \{s, sts, t\} = T$.

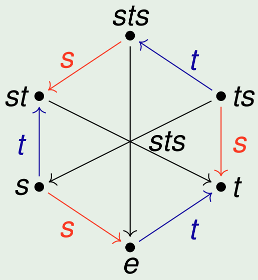


Twisted Bruhat

For $A \subseteq T$, flip the edges labelled with elements of A.

Example

Γ_{A_2} : s ●——● t $\quad N(st) = \{s, sts\} \subseteq \{s, sts, t\} = T.$



Twisted Bruhat

For $A \subseteq T$, flip the edges labelled with elements of A.

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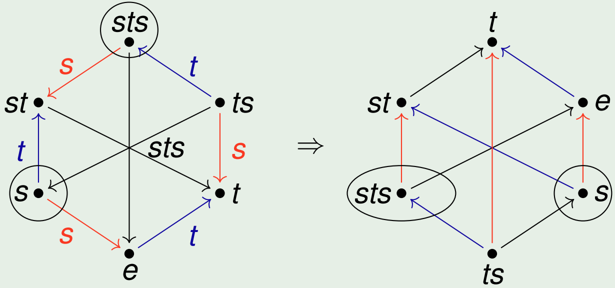
$N(st) = \{s, sts\} \subseteq \{s, sts, t\} = T.$



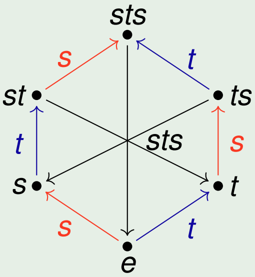
Twisted Bruhat order

Theorem (Edgar '09)

The twisted Bruhat graph is a partial order for $A \subseteq T$ if and only if A is an initial section.

Example (Non-example)

Γ_{A_2} :  $A = \{sts\} \subseteq \{s, sts, t\} = T$.



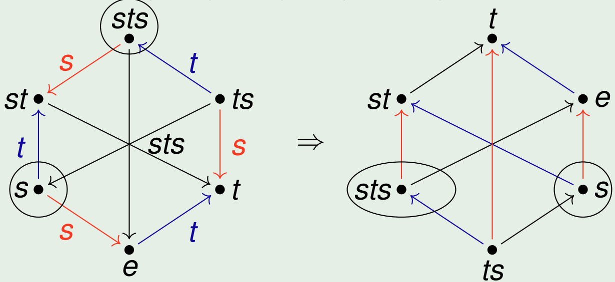
Twisted Bruhat order

Theorem (Edgar '09)

The twisted Bruhat graph is a partial order $\leq_A$ for $A \subseteq T$ if and only if A is an initial section.

Example

Γ_{A_2} : s —— t $\quad N(st) = \{s, sts\} \subseteq \{s, sts, t\} = T.$



Twisted Bruhat order

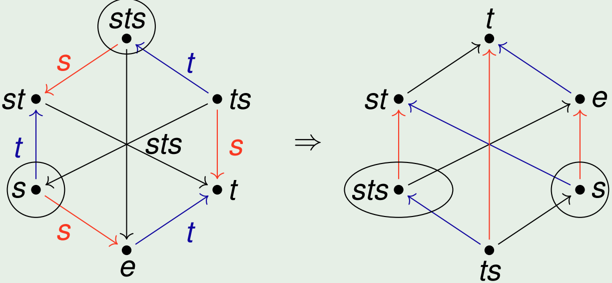
Theorem (D. 2025+)

For $t \in T$ and A an initial section:

$$t \leq_A e \iff t \in \overline{A}$$

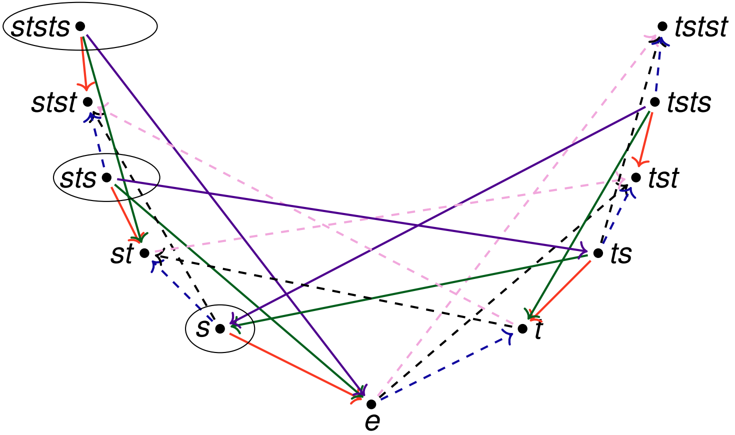
Example

Γ_{A_2} : s ——— t $N(st) = \{s, sts\} \subseteq \{s, sts, t\} = T$.

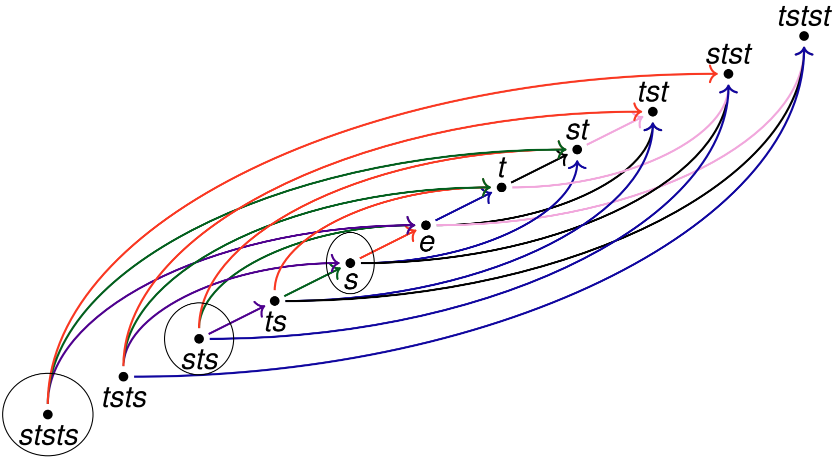


$\tilde{A}_1$ $\Gamma_{\tilde{A}_1}$: $s \overset{\infty}{\bullet\!\!-\!\!\bullet} t$ $\{w \in T \mid w \text{ begins with } s\}$

s : red *t* : blue *sts* : green *tst* : black *ststs* : purple *tstst* : pink



$\tilde{A}_1$ $\Gamma_{\tilde{A}_1}$: $s \overset{\infty}{—} t$ $\{w \in T \mid w \text{ begins with } s\}$



Bruhat order length

$$\ell_A(w) = \ell(w) - 2\left|N(w^{-1}) \cap A\right|.$$

Example

Γ_{A_2} : 

For $w = ts$ and $A = \{s, sts\} = N(st)$ we have

$$\begin{aligned} \ell_A(ts) &= \ell(ts) - 2|N(st) \cap A| \\ &= 2 - 2|\{s, sts\}| \\ &= 2 - 4 = -2 \end{aligned}$$

Theorem (D. 2025+)

For $t \in T$ and A an initial section:

$$\ell_A(t) \leq 0 \iff t \leq_A e$$

Initial section proof

Theorem (D. 2025+)

If $A \subseteq T$ is an initial section, then $\overline{A} = A$.

$$\begin{aligned} t \in A &\iff \ell_A(t) < 0 \qquad \text{(Dyer 1992)} \\ &\iff t \leq_A e \\ &\iff t \in \overline{A} \end{aligned}$$

Finite type preclosure

Z a set, $\mathrm{cl} : \mathcal{P}(Z) \to \mathcal{P}(Z)$ a preclosure.
cl is of *finite type* if

$$\mathrm{cl}(X) = \cup\{\mathrm{cl}(Y) \mid Y \subseteq X, |Y| \in \mathbb{N}\}$$

Note: $\mathrm{cl}^\infty(X) = \cup_{i \in \mathbb{N}} \mathrm{cl}^i(X)$ is a closure.

Theorem (D. 2025+)

Bruhat preclosure is of finite type.

Note: $\overline{A}^i$ is Bruhat preclosure performed i times:

$$\overline{A}^2 = \overline{\overline{A}} \quad \overline{A}^3 = \overline{\overline{\overline{A}}}$$

Corollary

*For *any* $A \subseteq T$, $\overline{A}^\infty$ is a closure.*

Dyer's conjecture

For $A \subseteq T$ and $u, v \in W$:

1. ~~$\overline{A}$ is a closure operator.~~ Kinda fixed - Initial sections + Infinite Bruhat closure
2. $\overline{N(u) \cup N(v)} = N(u \vee_R v)$.

Main theorem

Theorem (D. 2025+)

Let W be an arbitrary Coxeter group. For $u, v \in W$ such that $u \vee_R v$ exists, then

$$\overline{N(u) \cup N(v)}^{\infty} = N(u \vee_R v)$$

Corollary (D. 2025+)

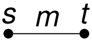
For W a finite Coxeter group, then for all $u, v \in W$

$$\overline{N(u) \cup N(v)}^{i} = N(u \vee_R v)$$

where $i \leq |T \backslash (N(u) \cup N(v))|$.

Progress - Dihedral

$\Gamma_{I_2(m)}$:



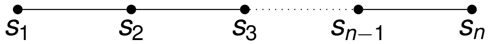
Theorem (Biagioli, Perrone 2025+, D. 2025+)

For W a dihedral group, for all $u, v \in W$,

$$\overline{N(u) \cup N(v)} = N(u \vee_R v).$$

Progress - Symmetric group - type A_n

Γ_{A_n} :



Theorem (Biagioli, Perrone 2025+)

For W a type A_n Coxeter group, for all $u, v \in W$

$$\overline{N(u) \cup N(v)} = N(u \vee_R v).$$

Theorem (D. 2025+)

For W a type A_n Coxeter group and $A \subseteq T$, $\overline{A}$ is a closure.

Progress - Other types

Finite types:

B_n Recently proved by Biagioli, Perrone 2026+

D_n Still open (Checked for $n \leq 5$)

F_4 Computer check by Biagioli, Perrone 2026+

H_3 Computer check by Biagioli, Perrone 2026+

H_4 Computer check by Dermenjian 2026+

E Still open (Check for E_6 in progress)

Infinite types:

- Everything open

Thank you