

The strong exchange property for Coxeter matroids

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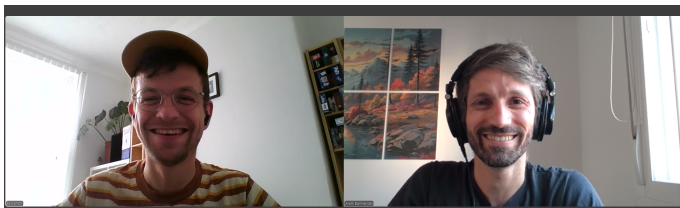
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Collaborators

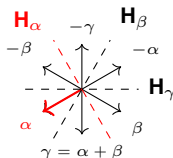


Roadmap

- 1 Coxeter matroids
- 2 Grassmannians and Tropicalization
- 3 Combinatorial Grassmannians of Coxeter matroids

Root systems

- $(V, \langle \cdot, \cdot \rangle)$ a real Euclidean vector space
- For $\alpha \in V \setminus \{0\}$, let s_α be its *reflection* with hyperplane $H_\alpha = \{x \in V \mid \langle x, \alpha \rangle = 0\}$.
- A *root system* is $\Phi \subset V$ such that for all $\alpha, \beta \in \Phi$.
 - $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$,
 - $s_\alpha(\Phi) = \Phi$,
 - $\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$
- $\Delta \subset \Phi$ a choice of linearly independent *simple roots*.



Weyl groups

- For a root system Φ and a fixed Δ the *Weyl group* of Φ is

$$W(\Phi) := \langle s_\alpha \mid \alpha \in \Delta \rangle.$$

- Finite types: $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$
- For $J \subset \Delta$, its *parabolic subgroup* $W_J \subset W(\Phi)$

$$W_J := \langle s_\beta \mid \beta \in J \rangle$$

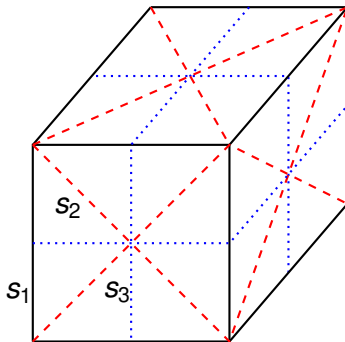
- *Maximal parabolic subgroup* W_J such that $J = \Delta \setminus \alpha$
- Quotients $W^J = W/W_J$
- *Parabolic cosets*: wW_J some representative for $w \in W^J$.

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Geometry of cosets

For $J \subseteq \Delta$, define x_J such that: $\langle x_J, \alpha \rangle \begin{cases} = 0 & \alpha \in J \\ < 0 & \alpha \notin J \end{cases}$

$$\{\mathbf{Cosets: } wW_J\} \longleftrightarrow \{\mathbf{Points: } w(x_J)\}$$

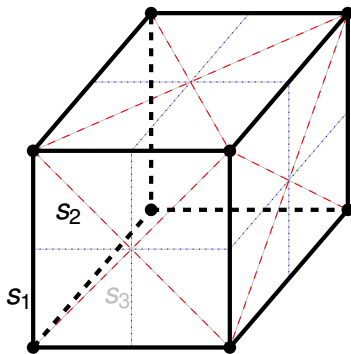


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 $J = \{\alpha_1, \alpha_2\}$



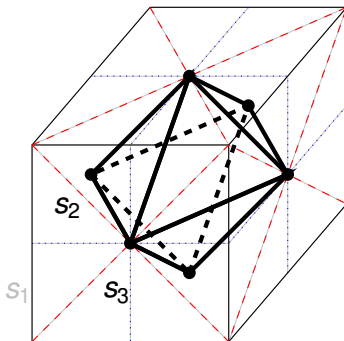
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Geometry of cosets

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{Cosets: wW_J } $\longleftrightarrow$ **{Points: $w(x_J)$ }**

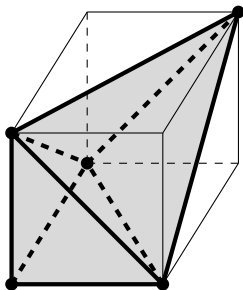
$$J = \{\alpha_2, \alpha_3\}$$



Coxeter matroids

Fix a root system Φ and a subset $J \subseteq \Delta$.

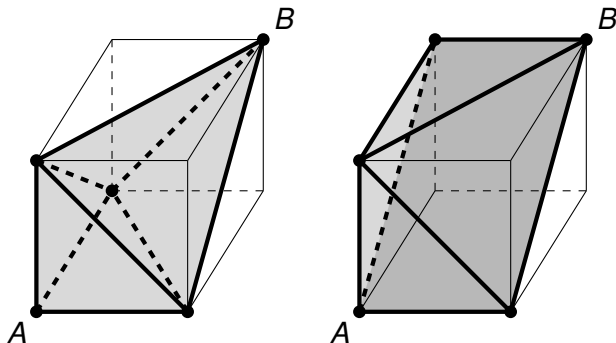
A subset $M \subseteq W^J$ is a *Coxeter matroid* if every edge of $\text{conv} \{w(x_J) \mid w \in M\}$ is parallel to some $\alpha \in \Phi$.



Strong Coxeter matroids

A Coxeter matroid $M \subseteq W^J$ is *strong* if it satisfies the *strong exchange property* for Coxeter matroids:

$$\forall v, w \in M, \exists H_\alpha \text{ separating } v, w \text{ such that } s_\alpha v, s_\alpha w \in M.$$



Question

How do we know if a given Coxeter matroid is strong by using only the polytope?

Grassmannians

Grassmannian: $\text{Gr}(k, n) := \{ k\text{-dim subspaces of } \mathbb{C}^n \}$

$$\text{Gr}(k, n) \hookrightarrow \text{Proj} \left(\bigwedge^k \mathbb{C}^n \right) \cong \mathbb{P}^{\binom{n}{k}-1}$$

$\text{Gr}(k, n)$ cut out by the *Plücker equations*:

$$I_{k,n} := \left\langle \sum_{j \in J \setminus I} (-1)^{\bullet} \cdot X_{I+j} \cdot X_{J-j} \mid I \in \binom{[n]}{k-1}, J \in \binom{[n]}{k+1} \right\rangle.$$

Example ($k = 2, n = 4$)

$\text{Gr}(2, 4) = V(I_{2,4}) \subseteq \mathbb{P}^5$ where

$$\begin{aligned} I_{2,4} &:= \langle X_{12} \cdot X_{34} - X_{13} \cdot X_{24} + X_{14} \cdot X_{23} \rangle \\ &\subseteq \mathbb{C}[X_{12}, X_{13}, X_{14}, X_{23}, X_{24}, X_{34}]. \end{aligned}$$

Tropicalization

Boolean semiring $\mathbb{B} := (\{0, 1\}, \oplus, \cdot)$ with $1 \oplus 1 = 1$.

A *tropical equation* f with support $A \subset \mathbb{N}^n$ is

$$f := \bigoplus_{a \in A} X^a \in \mathbb{B}[X_1, \dots, X_n], \quad X^a := X_1^{a_1} \cdots X_n^{a_n}$$

$$\text{trop}: \mathbb{C}[X] \longrightarrow \mathbb{B}[X]$$

$$\sum_{a \in A} z_a X^a \longmapsto \bigoplus_{a \in A} X^a$$

We say $v \in \mathbb{B}^n$ *satisfies* f if $|\{a \in A \mid v^a = 1\}| \neq 1$.

The combinatorial Grassmannian

Theorem (Speyer '08)

$M \subseteq \binom{[n]}{k}$ is a (strong) matroid if and only if v^M satisfies the tropicalized Plücker equations:

$$\text{trop}(\mathcal{P}_{I,J}) := \bigoplus_{j \in J \setminus I} X_{I+j} \cdot X_{J-j} \in \mathbb{B}[X] \quad \forall I \in \binom{[n]}{k-1}, J \in \binom{[n]}{k+1}.$$

Lie theory

Find tropical equations that cut out a combinatorial Grassmannian for Coxeter matroids.

- Φ root system
- $\mathbb{G}$ its complex simply connected Lie group
- $\mathfrak{g}$ its Lie algebra over complex v.s. V

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$$

- $V_{\pi} = (\pi, V)$ *representation of $\mathfrak{g}$* where $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$
 V decomposes into weight spaces $V = \bigoplus V(\lambda)$ where

$$V(\lambda) := \{v \in V \mid \pi(h) \cdot v = \langle \lambda, h \rangle v \ \forall h \in \mathfrak{h}\}, \quad \lambda \in \mathbb{R}\Phi$$

V_{π} a representation of $\mathfrak{g} \Rightarrow$ weights are invariant under $W(\Phi)$.

Grassmannians as group orbits

- Simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$,
- Maximal parabolic subgroups $W_{\Delta \setminus \alpha_i} \leq W(\Phi)$,
- Fundamental representations V_{λ_i} of $\mathfrak{g}$,
- Maximal parabolic subgroups $\mathbb{P}_{\alpha_i} = \text{stab}_{\mathbb{G}}(v_{\lambda_i}) \leq \mathbb{G}$.

$$\mathbb{G}/\mathbb{P}_{\alpha_i} \hookrightarrow \text{Proj}(V_{\lambda_i}) \quad g \mapsto g \cdot v_{\lambda_i}$$

Example ($\Phi = A_{n-1}$, $\mathbb{G} = SL_n(\mathbb{C})$)

The simple root α_k has fundamental representation $V_{\lambda_k} = \bigwedge^k \mathbb{C}^n$.

$$\text{Gr}(k, n) \cong SL_n(\mathbb{C}) / \text{stab}_{SL_n(\mathbb{C})}(U) \hookrightarrow \text{Proj} \left(\bigwedge^k \mathbb{C}^n \right)$$

where $U \subseteq \mathbb{C}^n$ standard k -dim linear space.

Quadric equations

$$\mathbb{G}/\mathbb{P}_{\alpha_i} \hookrightarrow \text{Proj}(V_{\lambda_i})$$

How do we characterise $\mathbb{G}/\mathbb{P}_{\alpha_i}$ via equations?

Theorem (Lichtenstein 1982)

The elements of $\text{Sym}^2(V_{\lambda}^{\vee})$ satisfying

$$\Omega(u \otimes v) - (2\lambda, 2\lambda + 2\rho)(u \otimes v) = 0$$

gives a system of quadratics cutting out $\mathbb{G}/\mathbb{P}$ inside $\text{Proj}(V_{\lambda})$.

This gives ‘equations’, but to get **equations**, need a canonical basis for V_{λ} .

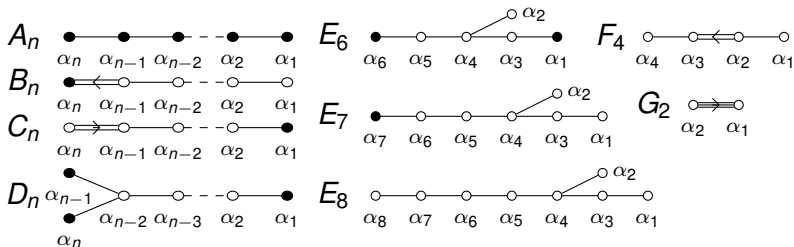
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Miniscule

Fundamental rep. V_{λ_i} is *minuscule* if $W(\Phi)$ acts transitively on the weights.

$\leadsto V_{\lambda_i} = \bigoplus V(\mu)$ decomposes into one-dim weight spaces.

We call $\mathbb{P}_{\alpha_i} \leq \mathbb{G}$ minuscule if V_{λ_i} minuscule.



Strong Coxeter matroids characterization

Theorem (Calvert, D., Fink, Smith '25+)

Let Φ be a root system with $\mathbb{G}$ simply connected complex Lie group and $\mathbb{P}_\alpha$ some minuscule parabolic subgroup. There exists a set of quadrics F cutting out $\mathbb{G}/\mathbb{P}_\alpha \subseteq \text{Proj}(V_\lambda)$ such that:

$M \subseteq W^{\Delta \setminus \alpha}$ a strong Coxeter matroid $\iff v^M$ satisfies $\text{trop}(F)$.

How to find F :

- Decompose $\text{Sym}^2(V_\lambda) = \bigoplus V_\mu$,
- Each $\mu \neq 2\lambda$ gives equations $\omega = 0$ for each basis vector $\omega \in V_\mu$,
- Rewrite ω in the V_λ basis to get an equation.

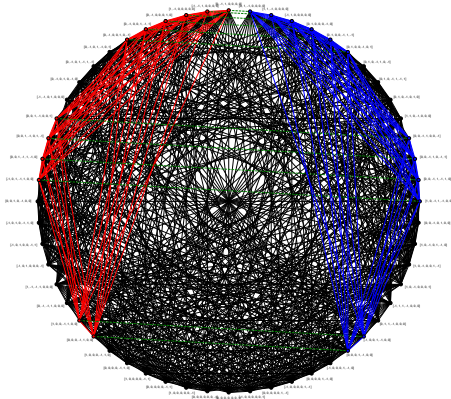
Computationally hard: type E_7

- According to the representation theory, we need 133 equations.
- V_μ is 133-dimensional, but decomposes into:
 - 126 one-dimensional weight spaces (indexed by roots of E_7)
 - 7-dimensional zero weight space (Cartan subalgebra of E_7)
- Use GAP to decompose our weight space into 1-dimensional components and get one equation for each.
- Use GAP to get 7 equations which have zero weight.

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Computationally hard: type E_7

GAP suggested the 7 equations we want are facet antipodes:



- After 6 months we could not prove this was true.
- Programmed it in SAGE, after 2 days: counterexample ▶

Computationally hard: type E_7

- Go back to drawing board.
- Use E_6 as a coset instead.
- This gives us one (unique) equation supported by the antipodes of E_7 .
- Use coset representatives to get remaining 6 (ensuring they are linearly independent as a set) - non-trivial conjugates.
- Proof falls naturally.
- Not so fun: coefficients are ± 1 and ± 3 .

Future Work

- How do we consider non-minuscule representations/parabolics?
- Any hope of quantifying non-strong Coxeter matroids?

Thanks