Facial Weak Order

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Joint work with: Christophe Hohlweg (LACIM) and Vincent Pilaud (CNRS & LIX)

Université du Québec à Montréal

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History and Background

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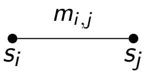
- *Finite Coxeter System* (W, S) such that

$$W := \langle s \in S \mid (s_i s_j)^{m_{i,j}} = e \text{ for } s_i, s_j \in S \rangle$$

where $m_{i,j} \in \mathbb{N}^\star$ and $m_{i,j} = 1$ only if $i = j$.

- A *Coxeter diagram* Γ_W for a Coxeter System (W, S) has S as a vertex set and an edge labelled $m_{i,j}$ when $m_{i,j} > 2$.



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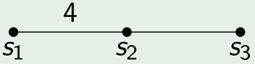
Example

$$W_{B_3} = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle$$

Γ_{B_3} :



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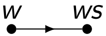
Let (W, S) be a Coxeter system.

- Let $w \in W$ such that $w = s_1 \ldots s_n$ for some $s_i \in S$. We say that w has *length* n, $\ell(w) = n$, if n is minimal.
- Let the *(right) weak order* be the order on the Cayley graph where 

and $\ell(w) < \ell(ws)$.
- For finite Coxeter systems, there exists a longest element in the weak order, $w_\circ$.

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Example

Let Γ_{A_2} : 

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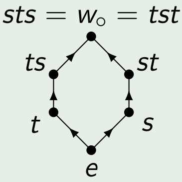

Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type A using inversion tables. They
  1. gave a local definition of this order using covers,
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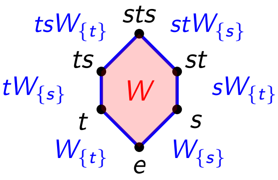
Parabolic Subgroups

Let $I \subseteq S$.

- $W_I = \langle I \rangle$ is the *standard parabolic subgroup* with long element denoted $w_{\circ,I}$.
- $W^I := \{w \in W \mid \ell(w) \leq \ell(ws), \text{ for all } s \in I\}$ is the set of minimal length coset representatives for W/W_I.
- Any element $w \in W$ admits a unique factorization $w = w^I \cdot w_I$ with $w^I \in W^I$ and $w_I \in W_I$.
- By convention in this talk xW_I means $x \in W^I$.
- *Coxeter complex* - $\mathcal{P}_W$ - the abstract simplicial complex whose faces are all the standard parabolic cosets of W.



Facial Weak Order

Definition (Krob et.al. [2001], Palacios, Ronco [2006])

The *(right) facial weak order* is the order $\leq_F$ on the Coxeter complex $\mathcal{P}_W$ defined by cover relations of two types:

$$(1) \qquad xW_I \lessdot xW_{I\cup\{s\}} \qquad \text{if } s \notin I \text{ and } x \in W^{I\cup\{s\}},$$

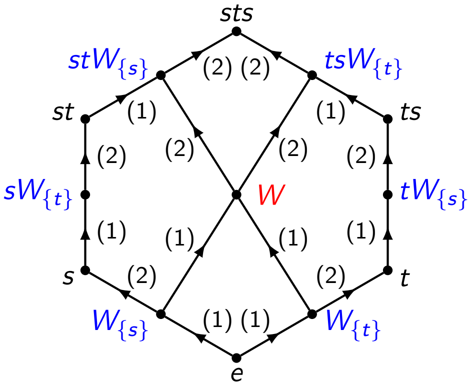
$$(2) \qquad xW_I \lessdot xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}}W_{I\smallsetminus\{s\}} \qquad \text{if } s \in I,$$

where $I \subseteq S$ and $x \in W^I$.

Facial weak order example

(1) $xW_I \lessdot xW_{I\cup\{s\}}$ if $s \notin I$ and $x \in W^{I\cup\{s\}}$

(2) $xW_I \lessdot xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}}W_{I\smallsetminus\{s\}}$ if $s \in I$



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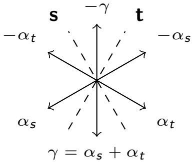
Root System

- Let $(V, \langle\cdot,\cdot\rangle)$ be a Euclidean space.
- Let W be a group generated by a set of reflections S. $W \hookrightarrow O(V)$ gives representation as a finite reflection group.
- The reflection associated to $\alpha \in V \setminus \{0\}$ is

$$s_\alpha(v) = v - \frac{2\langle v, \alpha\rangle}{||\alpha||^2}\alpha \quad (v \in V)$$



- A *root system* is $\Phi := \{\alpha \in V \mid s_\alpha \in W, ||\alpha|| = 1\}$
- We have $\Phi = \Phi^+ \sqcup \Phi^-$ decomposable into positive and negative roots.

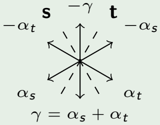
Inversion Sets

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

Let Γ_{A_2} : 

, with Φ given by the roots

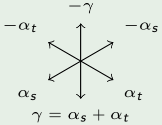


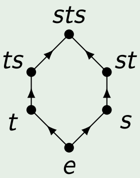
$$\begin{aligned}\mathbf{N}(ts) &= \Phi^+ \cap ts(\Phi^-)\\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\}\\ &= \{\alpha_t, \gamma\}\end{aligned}$$

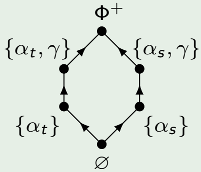
Weak order and Inversion sets

Given $w, u \in W$ then $w \leq_R u$ if and only if $\mathbf{N}(w) \subseteq \mathbf{N}(u)$.

Example

Let Γ_{A_2} :  , with Φ given by the roots 





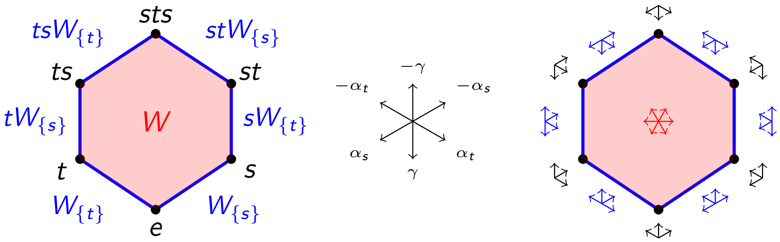
Root Inversion Set

Definition (Root Inversion Set)

Let xW_I be a standard parabolic coset. The *root inversion set* is the set

$$\mathbf{R}(xW_I) := x(\Phi^- \cup \Phi_I^+)$$

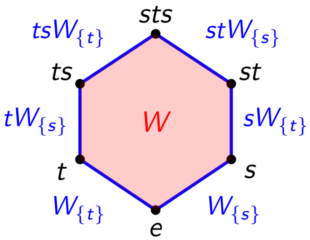
Note that $N(x) = \mathbf{R}(xW_\varnothing) \cap \Phi^+$.

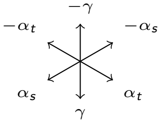


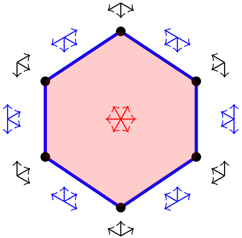
Root Inversion Set

Example

$$\begin{aligned}\mathbf{R}(sW_{\{t\}}) &= s(\Phi^- \cup \Phi^+_{\{t\}}) \\ &= s(\{-\alpha_s, -\alpha_t, -\gamma\} \cup \{\alpha_t\}) \\ &= \{\alpha_s, -\gamma, -\alpha_t, \gamma\}\end{aligned}$$







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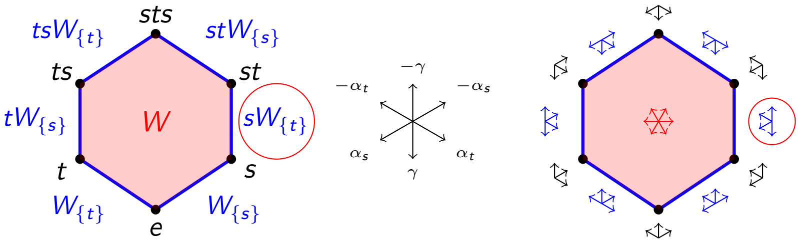


Equivalent definitions

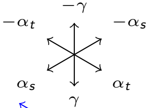
Theorem (D., Hohlweg, Pilaud [2016])

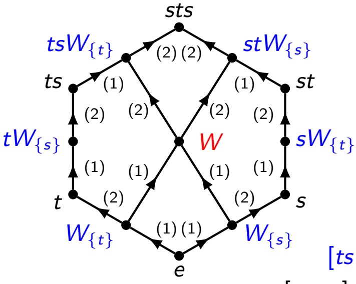
The following conditions are equivalent for two standard parabolic cosets xW_I and yW_J in the Coxeter complex $\mathcal{P}_W$

1. $xW_I \leq_F yW_J$
2. $\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$ *and* $\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$.
3. $x \leq_R y$ *and* $xw_{\circ,I} \leq_R yw_{\circ,J}$.

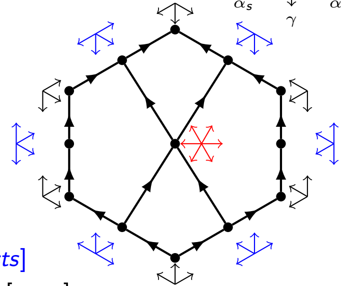
Remark Note that showing (1) ⇒ (3) and (3) ⇒ (2) is easy, but (2) ⇒ (1) is more difficult. We used induction on the symmetric difference between the root inversion sets for the proof.

Equivalence for type A_2 Coxeter System



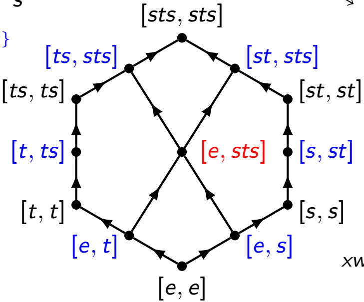


$xW_I \leq_F yW_J$



$\mathbf{R}(xW_I) \setminus \mathbf{R}(yW_J) \subseteq \mathbf{\Phi}^-$

$\mathbf{R}(yW_J) \setminus \mathbf{R}(xW_I) \subseteq \mathbf{\Phi}^+$



$x \leq_R y$

$xw_{\circ,I} \leq_R yw_{\circ,J}$

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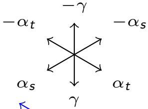
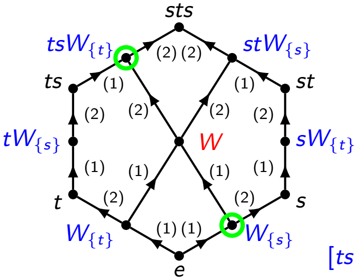
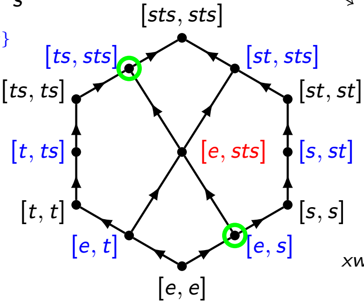
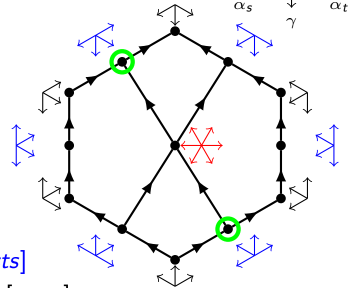




$xW_I \leq_F yW_J$





$\mathbf{R}(xW_I) \setminus \mathbf{R}(yW_J) \subseteq \Phi^-$

$\mathbf{R}(yW_J) \setminus \mathbf{R}(xW_I) \subseteq \Phi^+$

$x \leq_R y$

$xw_{\circ,I} \leq_R yw_{\circ,J}$

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Facial weak order lattice

Theorem (D., Hohlweg, Pilaud [2016])

The facial weak order $(\mathcal{P}_W, \leq_F)$ *is a lattice with the meet and join of two standard parabolic cosets* xW_I *and* yW_J *given by:*

$$xW_I \wedge yW_J = z_\wedge W_{K_\wedge},$$
$$xW_I \vee yW_J = z_\vee W_{K_\vee}.$$

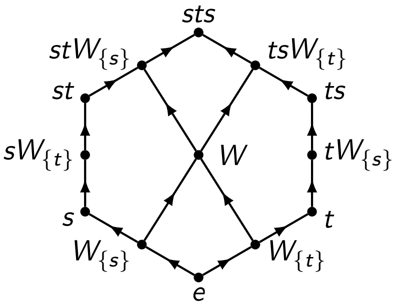
where,

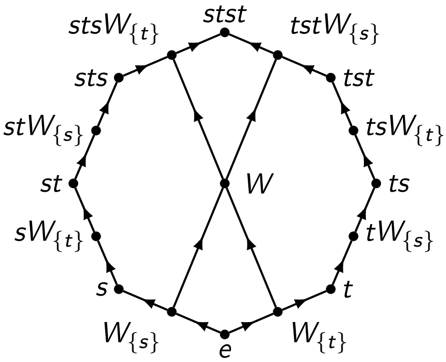
$$z_\wedge = x \wedge y \quad \text{and} \quad K_\wedge = D_L\big(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})\big), \text{ and}$$
$$z_\vee = xw_{\circ,I} \vee yw_{\circ,J} \quad \text{and} \quad K_\vee = D_L\big(z_\vee^{-1}(x \vee y)\big)$$

Corollary (D., Hohlweg, Pilaud [2016])

The weak order is a sublattice of the facial weak order lattice.

Example: A_2 and B_2





Example: A_2 and B_2

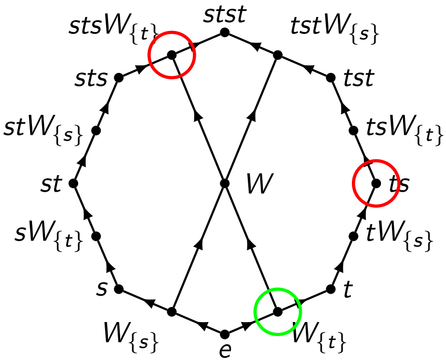
Example (Meet example)

Recall

$$xW_I \wedge yW_J = z_\wedge W_{K_\wedge}$$
$$\text{where} \quad z_\wedge = x \wedge y$$
$$K_\wedge = D_L(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J}))$$

We compute $ts \wedge stsW_{\{t\}}$.

$$z_\wedge = ts \wedge sts = e$$
$$K_\wedge = D_L(z_\wedge^{-1}(tsw_{\circ,\emptyset} \wedge stsw_{\circ,t}))$$
$$= D_L(e(ts \wedge stst))$$
$$= D_L(ts) = \{t\}.$$



Proof outline

Recall that $xW_I \leq_F yW_J \Leftrightarrow x \leq_r y$, and $xw_{\circ,I} \leq_R yw_{\circ,J}$.
We want to show that $xW_I \wedge yW_J = z_\wedge W_{K_\wedge}$ where $z_\wedge = x \wedge y$ and $K_\wedge = D_L\big(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})\big)$

- First we show that this element is in the Coxeter complex $z_\wedge \in W^{K_\wedge}$.
- We then show it's a lower bound: $x \wedge y \leq_R x, y$. Also, $w_{\circ,K_\wedge} \leq_R z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})$ implies $z_\wedge w_{\circ,K_\wedge} \leq_R xw_{\circ,I} \wedge yw_{\circ,J}$.
- Finally we show uniqueness by supposing there exists another element $zW_K \leq_F xW_I, yW_J$. Then we have $z \leq_R x \wedge y = z_\wedge$. Showing $zw_{\circ,K} \leq_R z_\wedge w_{\circ,K_\wedge}$ is done by looking at descents and the fact that $z \leq_R z_\wedge$.
- Join is found by an anti-automorphism.

Möbius function

Recall that the *Möbius function* of a poset $(P, \leq)$ is the function $\mu : P \times P \to \mathbb{Z}$ defined inductively by

$$\mu(p,q) := \begin{cases} 1 & \text{if } p = q, \\ -\sum_{p \leq r < q} \mu(p,r) & \text{if } p < q, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition (D., Hohlweg, Pilaud [2016])

The Möbius function of the facial weak order is given by

$$\mu(eW_{\varnothing}, yW_J) = \begin{cases} (-1)^{|J|}, & \textit{if } y = e, \\ 0, & \textit{otherwise.} \end{cases}$$

Quotients of the facial weak order

Lattice Congruences

Definition

A *lattice congruence* is an equivalence relation $\equiv$ on a lattice $(L, \leq)$ such that for each $x_1 \equiv x_2$ and $y_1 \equiv y_2$ then

1. $x_1 \wedge y_1 \equiv x_2 \wedge y_2$, and
2. $x_1 \vee y_1 \equiv x_2 \vee y_2$.

Theorem (D., Hohlweg, Pilaud [2016])

Given a lattice congruence $\equiv$ on $(W, \leq_R)$, the equivalence classes on $(\mathcal{P}_W, \leq_F)$ defined by

$$xW_I \equiv yW_J \iff x \equiv y \text{ and } xw_{\circ,I} \equiv yw_{\circ,J}$$

give us a lattice congruence.

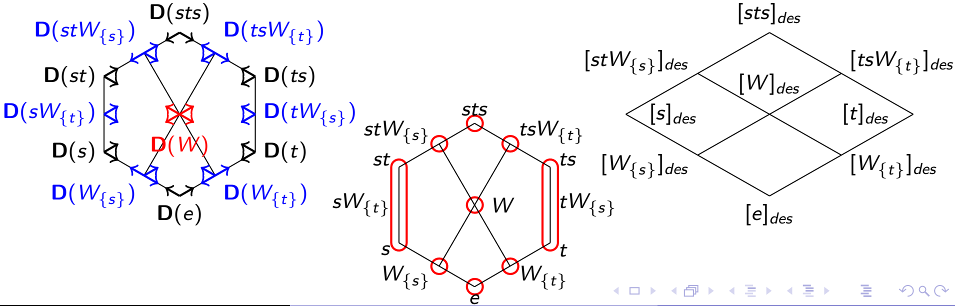
Facial Boolean Lattice

Corollary (D., Hohlweg, Pilaud [2016])

Let the (left) root descent set of a coset xW_I *be the set of roots*

$$\mathbf{D}(xW_I) := \mathbf{R}(xW_I) \cap \pm\Delta \subseteq \Phi.$$

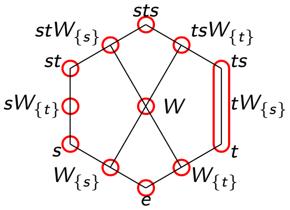
Let $xW_I \equiv^{\text{des}} yW_J$ *if and only if* $\mathbf{D}(xW_I) = \mathbf{D}(yW_J)$.

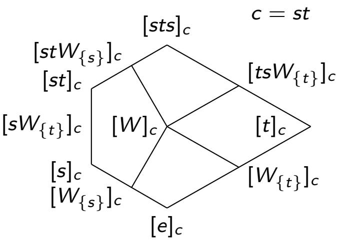


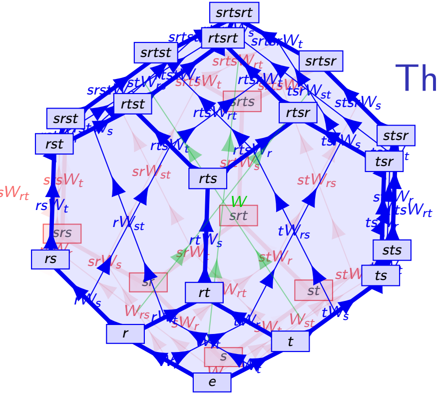
Facial Cambrian Lattice

Corollary (D., Hohlweg, Pilaud [2016])

Let c be any Coxeter element of W. Let $\equiv^c$ be the c-Cambrian congruence (see Reading [Cambrian Lattice, 2004]). Then let $xW_I \equiv^c yW_J \iff x \equiv^c y$ *and* $xw_{\circ,I} \equiv^c yw_{\circ,J}$.



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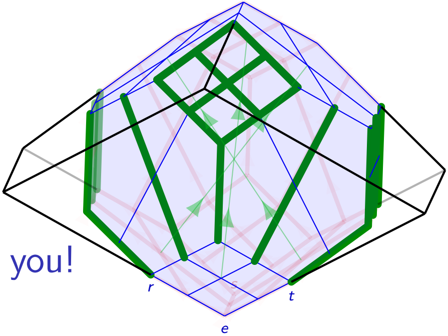
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