Facial Weak Order

Aram Dermenjian

Joint work with: Christophe Hohlweg (LACIM) and Vincent Pilaud (CNRS & LIX)

Université du Québec à Montréal

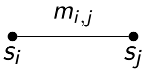
18 June 2017

History and Background

- *Finite Coxeter System (W, S)* such that

$$W := \langle s \in S \mid (s_i s_j)^{m_{i,j}} = e \text{ for } s_i, s_j \in S \rangle$$

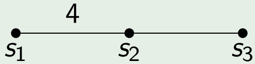
  where $m_{i,j} \in \mathbb{N}^{\star}$ and $m_{i,j} = 1$ only if $i = j$.
- A *Coxeter diagram* Γ_W for a Coxeter System (W, S) has S as a vertex set and an edge labelled $m_{i,j}$ when $m_{i,j} > 2$.



Example

$$W_{B_3} = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle$$

Γ_{B_3} :



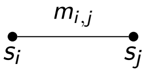
History and Background

- *Finite Coxeter System (W, S)* such that

$$W := \langle s \in S \mid (s_i s_j)^{m_{i,j}} = e \text{ for } s_i, s_j \in S \rangle$$

where $m_{i,j} \in \mathbb{N}^\star$ and $m_{i,j} = 1$ only if $i = j$.

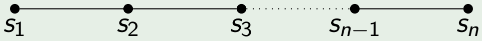
- A *Coxeter diagram* Γ_W for a Coxeter System (W, S) has S as a vertex set and an edge labelled $m_{i,j}$ when $m_{i,j} > 2$.



Example

$W_{A_n} = S_{n+1}$, symmetric group.

Γ_{A_n} :



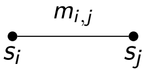
History and Background

- *Finite Coxeter System* (W, S) such that

$$W := \langle s \in S \mid (s_i s_j)^{m_{i,j}} = e \text{ for } s_i, s_j \in S \rangle$$

where $m_{i,j} \in \mathbb{N}^\star$ and $m_{i,j} = 1$ only if $i = j$.

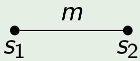
- A *Coxeter diagram* Γ_W for a Coxeter System (W, S) has S as a vertex set and an edge labelled $m_{i,j}$ when $m_{i,j} > 2$.



Example

$W_{I_2(m)} = \mathcal{D}(m)$, dihedral group of order $2m$.

$\Gamma_{I_2(m)}$:

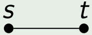


History and Background

Let (W, S) be a Coxeter system.

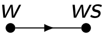
- Let $w \in W$ such that $w = s_1 \ldots s_n$ for some $s_i \in S$. We say that w has *length* n, $\ell(w) = n$, if n is minimal.

Example

Let Γ_{A_2} : 

.

$\ell(stst) = 2$ as $stst = tstt = ts$.

- Let the *(right) weak order* be the order on the Cayley graph where 

and $\ell(w) <_R \ell(ws)$.

History and Background

Theorem (Björner [1984])

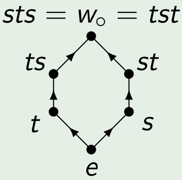
Let (W, S) be a finite Coxeter system. The weak order is a lattice graded by length.

- For finite Coxeter systems, there exists a longest element in the weak order, $w_\circ$.

Example

Let Γ_{A_2} : 

.



Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type A using inversion tables. They
  1. gave a local definition of this order using covers,
  2. gave a global definition of this order combinatorially, and
  3. showed that the poset for this order is a lattice.
- In 2006, Ronco and Palacios extended this new order to Coxeter groups of all types using cover relations.

Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type A using inversion tables. They
  1. gave a local definition of this order using covers, ✓
  2. gave a global definition of this order combinatorially, and
  3. showed that the poset for this order is a lattice.
- In 2006, Ronco and Palacios extended this new order to Coxeter groups of all types using cover relations.
- Problems: Can we find a global definition for this poset, and is it a lattice?

Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type A using inversion tables. They
  1. gave a local definition of this order using covers, ✓
  2. gave a global definition of this order combinatorially, and ✓
  3. showed that the poset for this order is a lattice. ✓
- In 2006, Ronco and Palacios extended this new order to Coxeter groups of all types using cover relations.
- Problems: Can we find a global definition for this poset, and is it a lattice?

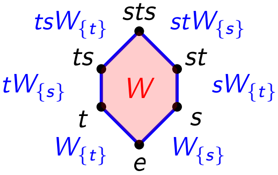
Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type A using inversion tables. They
    1. gave a local definition of this order using covers, ✓
    2. gave a global definition of this order combinatorially, and ✓
    3. showed that the poset for this order is a lattice. ✓
- In 2006, Ronco and Palacios extended this new order to Coxeter groups of all types using cover relations.
- Problems: Can we find a global definition for this poset, and is it a lattice?

Parabolic Subgroups

Let $I \subseteq S$.

- $W_I = \langle I \rangle$ is the *standard parabolic subgroup* with long element denoted $w_{\circ,I}$.
- $W^I := \{w \in W \mid \ell(w) \leq \ell(ws), \text{ for all } s \in I\}$ is the set of minimal length coset representatives for W/W_I.
- Any element $w \in W$ admits a unique factorization $w = w^I \cdot w_I$ with $w^I \in W^I$ and $w_I \in W_I$.
- By convention in this talk xW_I means $x \in W^I$.
- *Coxeter complex* - $\mathcal{P}_W$ - the abstract simplicial complex whose faces are all the standard parabolic cosets of W.



Facial Weak Order

Let (W, S) be a finite Coxeter system.

Definition (Krob et.al. [2001, type A], Palacios, Ronco [2006])

The *(right) facial weak order* is the order $\leq_F$ on the Coxeter complex $\mathcal{P}_W$ defined by cover relations of two types:

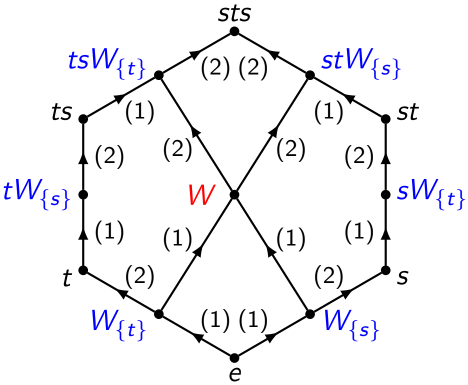
$$(1) \qquad xW_I \lessdot xW_{I\cup\{s\}} \qquad \text{if } s \notin I \text{ and } x \in W^{I\cup\{s\}},$$

$$(2) \qquad xW_I \lessdot xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}}W_{I\smallsetminus\{s\}} \qquad \text{if } s \in I,$$

where $I \subseteq S$ and $x \in W^I$.

Facial weak order example

(1) $xW_I \lessdot xW_{I\cup\{s\}}$ if $s \notin I$ and $x \in W^{I\cup\{s\}}$
(2) $xW_I \lessdot xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}}W_{I\smallsetminus\{s\}}$ if $s \in I$



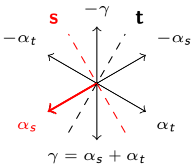
Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type A using inversion tables. They
  1. gave a local definition of this order using covers, ✓
  2. gave a global definition of this order combinatorially, and ✓
  3. showed that the poset for this order is a lattice. ✓
- In 2006, Ronco and Palacios extended this new order to Coxeter groups of all types using cover relations.
- Problems: Can we find a global definition for this poset, and is it a lattice?

Root System

- Let $(V, \langle\cdot,\cdot\rangle)$ be a real Euclidean space.
- Let W be a group generated by a set of reflections S. $W \hookrightarrow O(V)$ gives representation as a finite reflection group.
- The reflection associated to $\alpha \in V \backslash \{0\}$ is

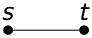
$$s_\alpha(v) = v - \frac{2\langle v, \alpha\rangle}{||\alpha||^2}\alpha \quad (v \in V)$$

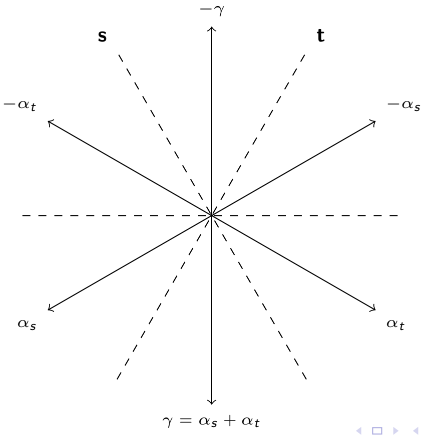


- A *root system* is $\Phi := \{\alpha \in V \mid s_\alpha \in W, ||\alpha|| = 1\}$
- We have $\Phi = \Phi^+ \sqcup \Phi^-$ decomposable into positive and negative roots.

Relationship between Root Systems and Coxeter Systems

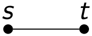
$W_{A_2} = \langle s, t \mid s^2 = t^2 = (st)^3 = e \rangle$ Γ_{A_2} :



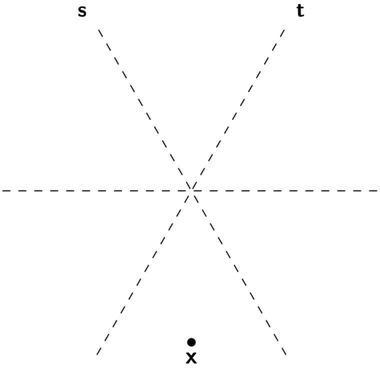


Relationship between Root Systems and Coxeter Systems

$W_{A_2} = \langle s, t \mid s^2 = t^2 = (st)^3 = e \rangle$ Γ_{A_2} :

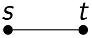


$\text{Perm}(W) = \{w(x) \mid w \in W\}$

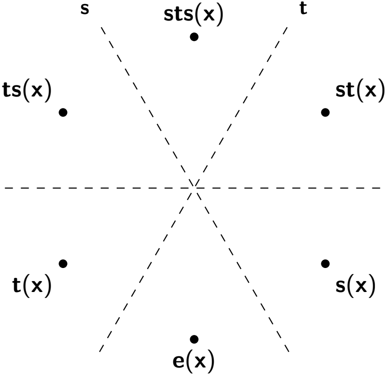


Relationship between Root Systems and Coxeter Systems

$W_{A_2} = \langle s, t \mid s^2 = t^2 = (st)^3 = e \rangle$ Γ_{A_2} :

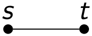


$\mathrm{Perm}(W) = \{w(x) \mid w \in W\}$

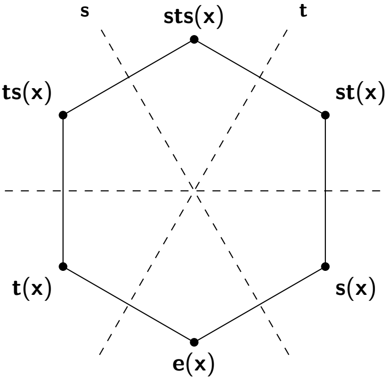


Relationship between Root Systems and Coxeter Systems

$W_{A_2} = \langle s, t \mid s^2 = t^2 = (st)^3 = e \rangle$ Γ_{A_2} :

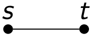


$\mathrm{Perm}(W) = \{w(x) \mid w \in W\}$

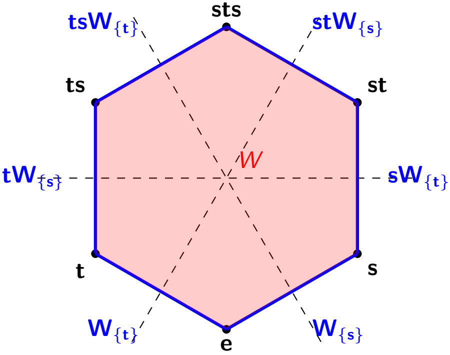


Relationship between Root Systems and Coxeter Systems

$W_{A_2} = \langle s, t \mid s^2 = t^2 = (st)^3 = e \rangle$ Γ_{A_2} :



$\mathrm{Perm}(W) = \{w(x) \mid w \in W\}$

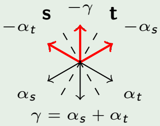


Inversion Sets

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

Let Γ_{A_2} : 

, with Φ given by the roots 

$$\begin{aligned} \mathbf{N}(ts) &= \Phi^+ \cap ts(\Phi^-) \\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\} \\ &= \{\alpha_t, \gamma\} \end{aligned}$$

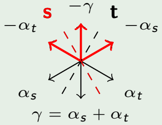
Inversion Sets

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

Let Γ_{A_2} : 

, with Φ given by the roots



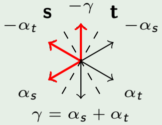
$$\begin{aligned}\mathbf{N}(ts) &= \Phi^+ \cap ts(\Phi^-)\\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\}\\ &= \{\alpha_t, \gamma\}\end{aligned}$$

Inversion Sets

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

Let Γ_{A_2} : s ●——● t , with Φ given by the roots



$$\begin{aligned}\mathbf{N}(ts) &= \Phi^+ \cap ts(\Phi^-)\\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\}\\ &= \{\alpha_t, \gamma\}\end{aligned}$$

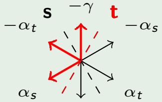
Inversion Sets

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

Let Γ_{A_2} : 

, with Φ given by the roots



$\gamma = \alpha_s + \alpha_t$

$$\begin{aligned} \mathbf{N}(ts) &= \Phi^+ \cap ts(\Phi^-) \\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\} \\ &= \{\alpha_t, \gamma\} \end{aligned}$$

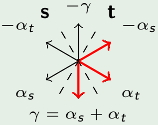
Inversion Sets

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

Let Γ_{A_2} : 

, with Φ given by the roots



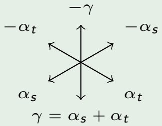
$$\begin{aligned}\mathbf{N}(ts) &= \Phi^+ \cap ts(\Phi^-) \\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\} \\ &= \{\alpha_t, \gamma\}\end{aligned}$$

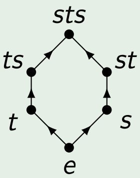
Weak order and Inversion sets

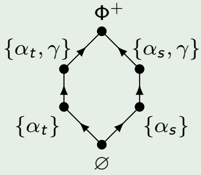
Given $w, u \in W$ then $w \leq_R u$ if and only if $\mathbf{N}(w) \subseteq \mathbf{N}(u)$.

Example

Let Γ_{A_2} : 

, with Φ given by the roots 





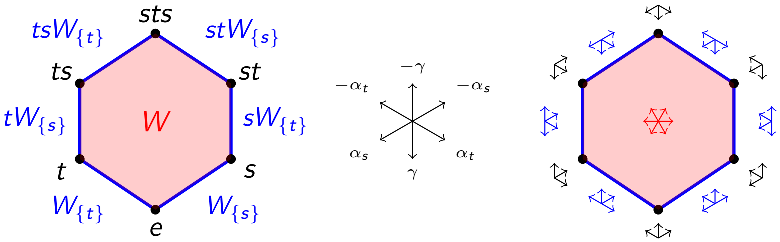
Root Inversion Set

Definition (Root Inversion Set)

Let xW_I be a standard parabolic coset. The *root inversion set* is the set

$$\mathbf{R}(xW_I) := x(\Phi^- \cup \Phi_I^+)$$

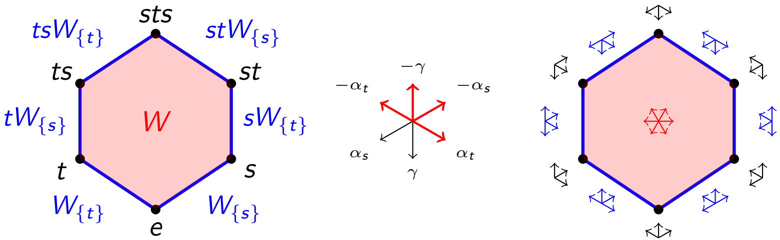
Note that $N(x) = \mathbf{R}(xW_\varnothing) \cap \Phi^+$.



Root Inversion Set

Example

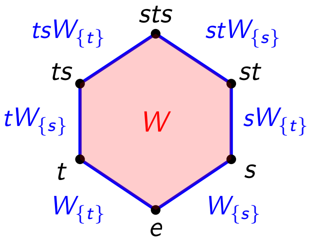
$$\mathbf{R}(sW_{\{t\}}) = s(\Phi^- \cup \Phi^+_{\{t\}})$$
$$= s(\{-\alpha_s, -\alpha_t, -\gamma\} \cup \{\alpha_t\})$$
$$= \{\alpha_s, -\gamma, -\alpha_t, \gamma\}$$

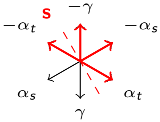


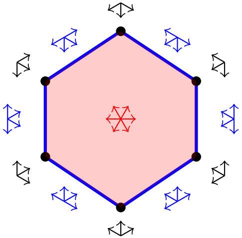
Root Inversion Set

Example

$$\mathbf{R}(sW_{\{t\}}) = s(\Phi^- \cup \Phi^+_{\{t\}})$$
$$= s(\{-\alpha_s, -\alpha_t, -\gamma\} \cup \{\alpha_t\})$$
$$= \{\alpha_s, -\gamma, -\alpha_t, \gamma\}$$



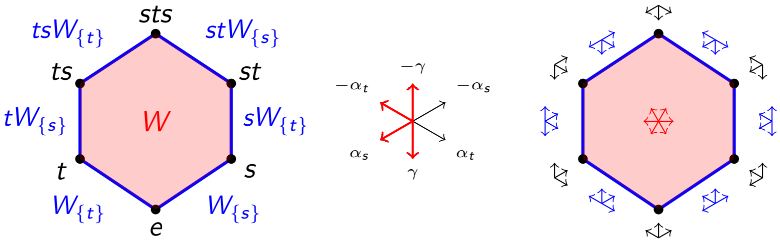




Root Inversion Set

Example

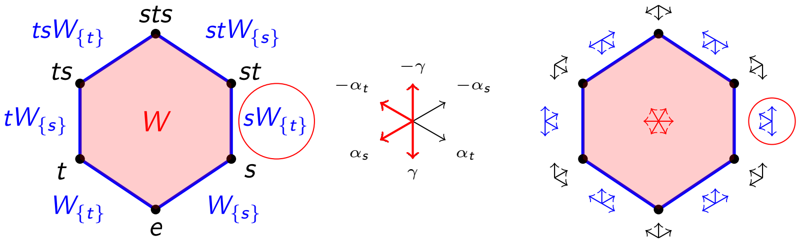
$$\mathbf{R}(sW_{\{t\}}) = s(\Phi^- \cup \Phi^+_{\{t\}})$$
$$= s(\{-\alpha_s, -\alpha_t, -\gamma\} \cup \{\alpha_t\})$$
$$= \{\alpha_s, -\gamma, -\alpha_t, \gamma\}$$



Root Inversion Set

Example

$$\begin{aligned}\mathbf{R}(sW_{\{t\}}) &= s(\Phi^- \cup \Phi^+_{\{t\}}) \\ &= s(\{-\alpha_s, -\alpha_t, -\gamma\} \cup \{\alpha_t\}) \\ &= \{\alpha_s, -\gamma, -\alpha_t, \gamma\}\end{aligned}$$



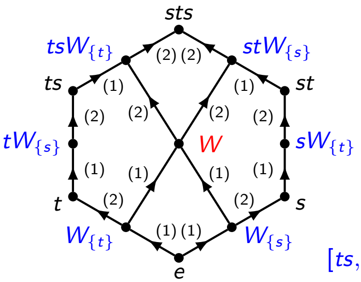
Equivalent definitions

Theorem (D., Hohlweg, Pilaud [2016])

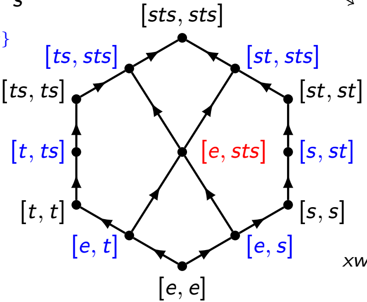
The following conditions are equivalent for two standard parabolic cosets xW_I and yW_J in the Coxeter complex $\mathcal{P}_W$

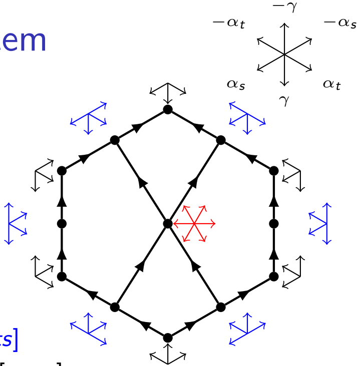
1. $xW_I \leq_F yW_J$
2. $\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$ *and* $\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$.
3. $x \leq_R y$ *and* $xw_{\circ,I} \leq_R yw_{\circ,J}$.

Equivalence for type A_2 Coxeter System



$xW_I \leq_F yW_J$





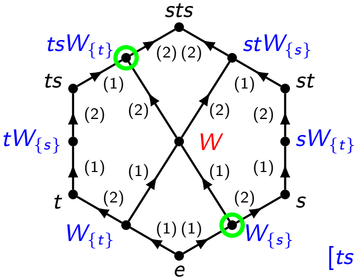
$\mathbf{R}(xW_I) \setminus \mathbf{R}(yW_J) \subseteq \Phi^-$

$\mathbf{R}(yW_J) \setminus \mathbf{R}(xW_I) \subseteq \Phi^+$

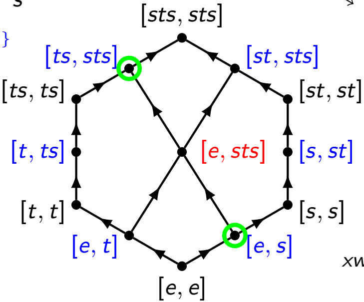
$x \leq_R y$

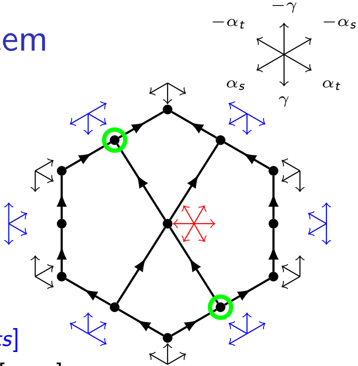
$xw_{\circ,I} \leq_R yw_{\circ,J}$

Equivalence for type A_2 Coxeter System



$xW_I \leq_F yW_J$



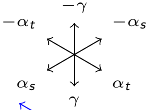


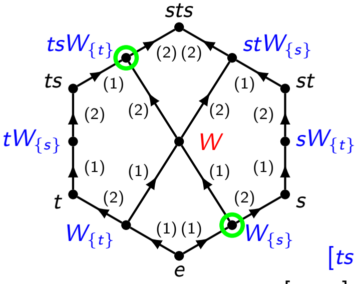
$\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$
$\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$

$x \leq_R y$

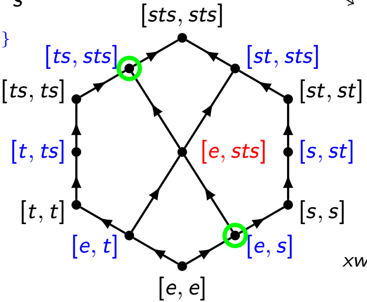
$xw_{\circ,I} \leq_R yw_{\circ,J}$

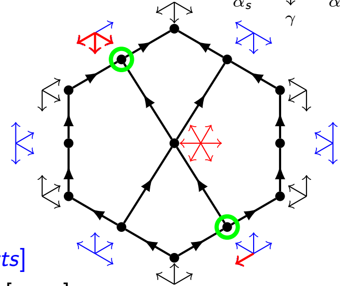
Equivalence for type A_2 Coxeter System





$xW_I \leq_F yW_J$





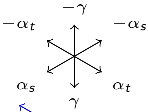
$\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \mathbf{\Phi}^-$

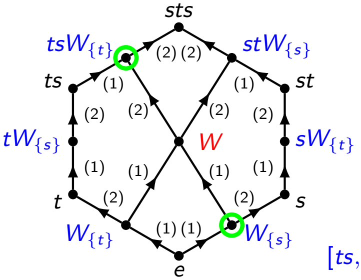
$\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \mathbf{\Phi}^+$

$x \leq_R y$

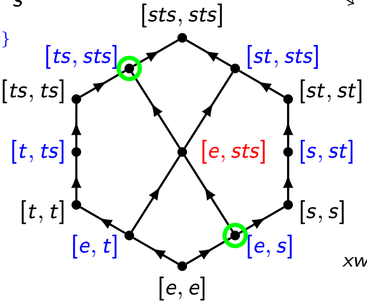
$xw_{\circ,I} \leq_R yw_{\circ,J}$

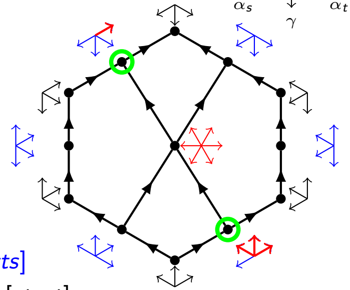
Equivalence for type A_2 Coxeter System





$xW_I \leq_F yW_J$





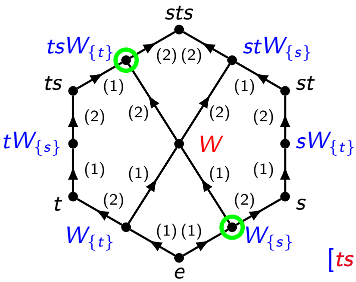
$\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$

$\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$

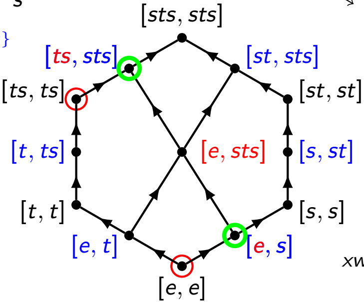
$x \leq_R y$

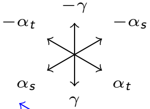
$xw_{\circ,I} \leq_R yw_{\circ,J}$

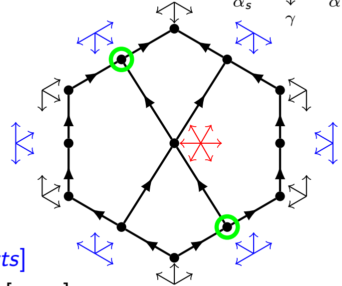
Equivalence for type A_2 Coxeter System



$xW_I \leq_F yW_J$





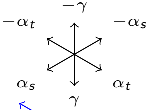


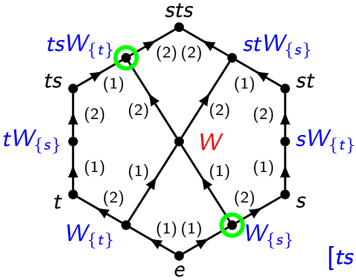
$\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$
$\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$

$x \leq_R y$

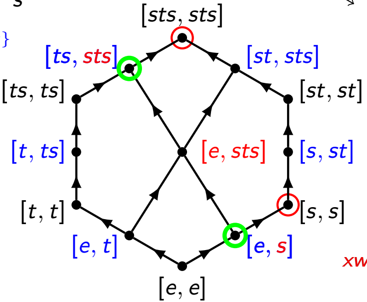
$xw_{\circ,I} \leq_R yw_{\circ,J}$

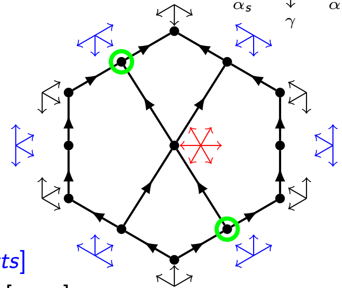
Equivalence for type A_2 Coxeter System





$xW_I \leq_F yW_J$





$\mathbf{R}(xW_I) \setminus \mathbf{R}(yW_J) \subseteq \Phi^-$

$\mathbf{R}(yW_J) \setminus \mathbf{R}(xW_I) \subseteq \Phi^+$

$x \leq_R y$

$xw_{\circ,I} \leq_R yw_{\circ,J}$

Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type A using inversion tables. They
  1. gave a local definition of this order using covers, ✓
  2. gave a global definition of this order combinatorially, and ✓
  3. showed that the poset for this order is a lattice. ✓
- In 2006, Ronco and Palacios extended this new order to Coxeter groups of all types using cover relations.
- Problems: Can we find a global definition for this poset, and is it a lattice?

Facial weak order lattice

Theorem (D., Hohlweg, Pilaud [2016])

The facial weak order $(\mathcal{P}_W, \leq_F)$ *is a lattice with the meet and join of two standard parabolic cosets* xW_I *and* yW_J *given by:*

$$xW_I \wedge yW_J = z_\wedge W_{K_\wedge},$$
$$xW_I \vee yW_J = z_\vee W_{K_\vee}.$$

where,

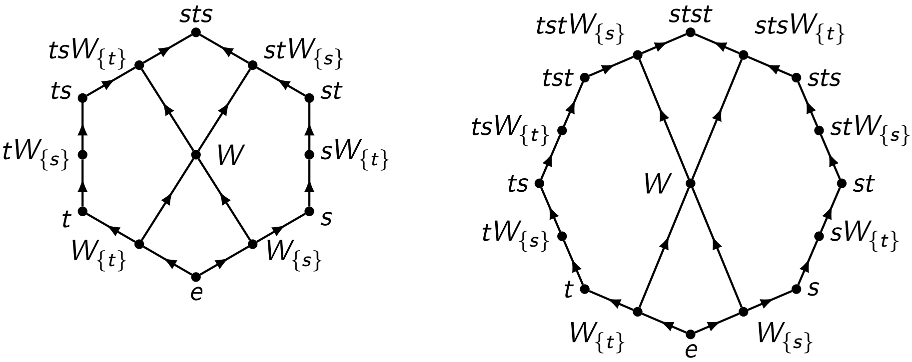
$z_\wedge = x \wedge y$ *and* $K_\wedge = D_L(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J}))$, *and*

$z_\vee = xw_{\circ,I} \vee yw_{\circ,J}$ *and* $K_\vee = D_L(z_\vee^{-1}(x \vee y))$

Corollary (D., Hohlweg, Pilaud [2016])

The weak order is a sublattice of the facial weak order lattice.

Example: A_2 and B_2



Example: A_2 and B_2

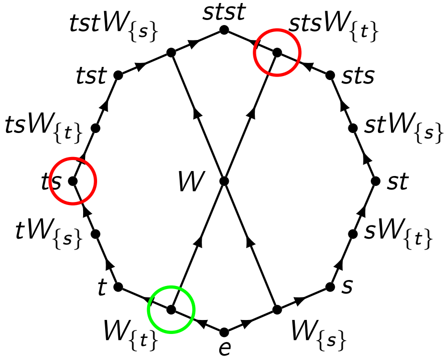
Example (Meet example)

Recall

$$xW_I \wedge yW_J = z_\wedge W_{K_\wedge}$$
$$\text{where} \quad z_\wedge = x \wedge y$$
$$K_\wedge = D_L(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J}))$$

We compute $ts \wedge stsW_{\{t\}}$.

$$z_\wedge = ts \wedge sts = e$$
$$K_\wedge = D_L(z_\wedge^{-1}(tsw_{\circ,\emptyset} \wedge stsw_{\circ,t}))$$
$$= D_L(e(ts \wedge stst))$$
$$= D_L(ts) = \{t\}.$$



Möbius function

Recall that the *Möbius function* of a poset $(P, \leq)$ is the function $\mu : P \times P \to \mathbb{Z}$ defined inductively by

$$\mu(p,q) := \begin{cases} 1 & \text{if } p = q, \\ -\sum_{p \leq r < q} \mu(p,r) & \text{if } p < q, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition (D., Hohlweg, Pilaud [2016])

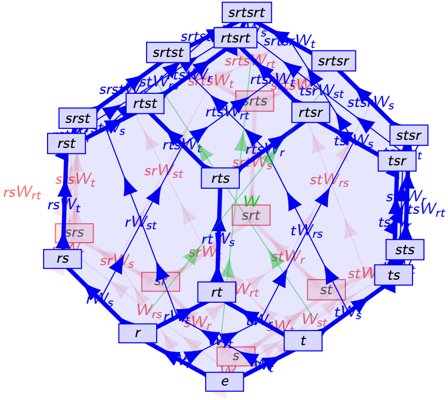
The Möbius function of the facial weak order is given by

$$\mu(eW_{\varnothing}, yW_J) = \begin{cases} (-1)^{|J|}, & \text{if } y = e, \\ 0, & \text{otherwise.} \end{cases}$$

Further Works

- Work with Thomas McConville (MIT) to extend the facial weak order to oriented matroids.
- Can we extend the weak order to other objects?

Thank you!


srtsrt
rtsrt
srtst
srtsr
rtst
rtsr
srst
stsr
rst
tsr
rts
srt
srs
sts
rs
ts
sr
st
rt
r
t
s
e
rsW_rt
srW_st
stW_rs
rW_st
tW_rs