Facial Weak Order

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Joint work with: Christophe Hohlweg (LACIM) and Vincent Pilaud (CNRS & LIX)

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History and Background

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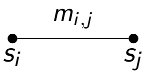
- *Finite Coxeter System* (W, S) such that

$$W := \langle s \in S \mid (s_i s_j)^{m_{i,j}} = e \text{ for } s_i, s_j \in S \rangle$$

where $m_{i,j} \in \mathbb{N}^\star$ and $m_{i,j} = 1$ only if $i = j$.

- A *Coxeter diagram* Γ_W for a Coxeter System (W, S) has S as a vertex set and an edge labelled $m_{i,j}$ when $m_{i,j} > 2$.



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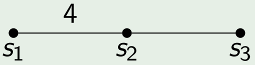
Example

$$W_{B_3} = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle$$

Γ_{B_3} :



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Let (W, S) be a Coxeter system.

- Let $w \in W$ such that $w = s_1 \dots s_n$ for some $s_i \in S$. We say that w has *length* n, $\ell(w) = n$, if n is minimal.
- Let the *(right) weak order* be the order on the Cayley graph where $\overset{w}{\bullet} \longrightarrow \overset{ws}{\bullet}$ and $\ell(w) < \ell(ws)$.
- For finite Coxeter systems, there exists a longest element in the weak order, $w_\circ$.

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Example

Let Γ_{A_2} : 

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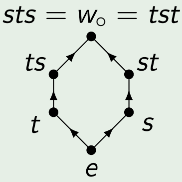

Motivation

- In 2001, Krob, Latapy, Novelli, Phan, and Schwer extended the weak order to an order on all faces for type A using inversion tables. They
  1. gave a local definition of this order using covers,
  2. gave a global definition of this order combinatorially, and
  3. showed that the poset for this order is a lattice.
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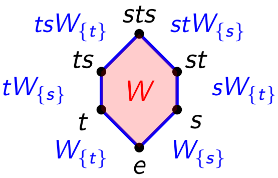
Parabolic Subgroups

Let $I \subseteq S$.

- $W_I = \langle I \rangle$ is the *standard parabolic subgroup* with long element denoted $w_{\circ,I}$.
- $W^I := \{w \in W \mid \ell(w) \leq \ell(ws), \text{ for all } s \in I\}$ is the set of minimal length coset representatives for W/W_I.
- Any element $w \in W$ admits a unique factorization $w = w^I \cdot w_I$ with $w^I \in W^I$ and $w_I \in W_I$.
- By convention in this talk xW_I means $x \in W^I$.
- *Coxeter complex* - $\mathcal{P}_W$ - the abstract simplicial complex whose faces are all the standard parabolic cosets of W.



Facial Weak Order

Definition (Krob et.al. [2001], Palacios, Ronco [2006])

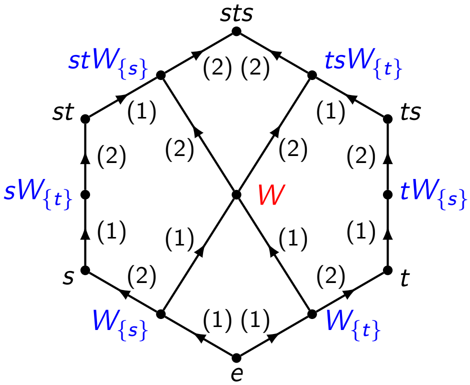
The *(right) facial weak order* is the order $\leq_F$ on the Coxeter complex $\mathcal{P}_W$ defined by cover relations of two types:

$$
\begin{aligned}
(1) \qquad & xW_I \lessdot xW_{I\cup\{s\}} \qquad && \text{if } s \notin I \text{ and } x \in W^{I\cup\{s\}}, \\
(2) \qquad & xW_I \lessdot xw_{\circ,I}w_{\circ,I\smallsetminus\{s\}}W_{I\smallsetminus\{s\}} \qquad && \text{if } s \in I,
\end{aligned}
$$

where $I \subseteq S$ and $x \in W^I$.

Facial weak order example

(1) $xW_I \lessdot xW_{I \cup \{s\}}$ if $s \notin I$ and $x \in W^{I \cup \{s\}}$
(2) $xW_I \lessdot xw_{\circ,I}w_{\circ,I \setminus \{s\}}W_{I \setminus \{s\}}$ if $s \in I$



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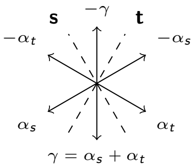
Root System

- Let $(V, \langle\cdot,\cdot\rangle)$ be a Euclidean space.
- Let W be a group generated by a set of reflections S. $W \hookrightarrow O(V)$ gives representation as a finite reflection group.
- The reflection associated to $\alpha \in V\backslash\{0\}$ is

$$s_\alpha(v) = v - \frac{2\langle v, \alpha\rangle}{||\alpha||^2}\alpha \quad (v \in V)$$



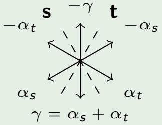
- A *root system* is $\Phi := \{\alpha \in V \mid s_\alpha \in W,\ ||\alpha|| = 1\}$
- We have $\Phi = \Phi^+ \sqcup \Phi^-$ decomposable into positive and negative roots.

Inversion Sets

Let (W, S) be a Coxeter system.
Define *(left) inversion sets* as the set $\mathbf{N}(w) := \Phi^+ \cap w(\Phi^-)$.

Example

Let Γ_{A_2} : $\overset{s}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{t}{\bullet}$, with Φ given by the roots



$$\begin{aligned}\mathbf{N}(ts) &= \Phi^+ \cap ts(\Phi^-)\\ &= \Phi^+ \cap \{\alpha_t, \gamma, -\alpha_s\}\\ &= \{\alpha_t, \gamma\}\end{aligned}$$

Weak order and Inversion sets

Given $w, u \in W$ then $w \leq_R u$ if and only if $\mathbf{N}(w) \subseteq \mathbf{N}(u)$.

Example

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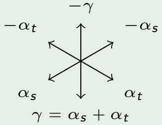
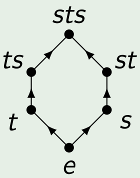
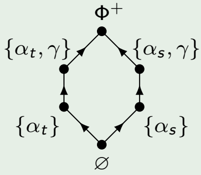






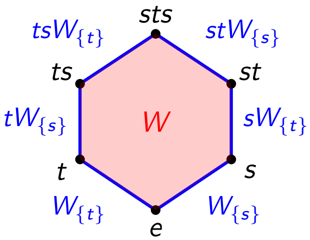
Root Inversion Set

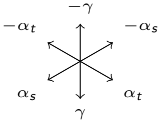
Definition (Root Inversion Set)

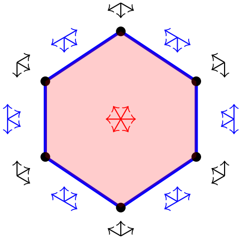
Let xW_I be a standard parabolic coset. The *root inversion set* is the set

$$\mathbf{R}(xW_I) := x(\Phi^- \cup \Phi_I^+)$$

Note that $N(x) = \mathbf{R}(xW_\varnothing) \cap \Phi^+$.



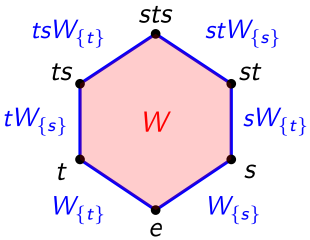


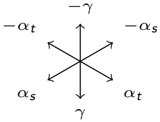


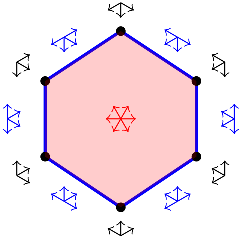
Root Inversion Set

Example

$$\begin{aligned}\mathbf{R}(sW_{\{t\}}) &= s(\Phi^{-} \cup \Phi^{+}_{\{t\}}) \\ &= s(\{-\alpha_s, -\alpha_t, -\gamma\} \cup \{\alpha_t\}) \\ &= \{\alpha_s, -\gamma, -\alpha_t, \gamma\}\end{aligned}$$







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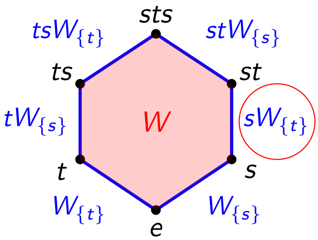
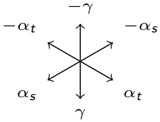
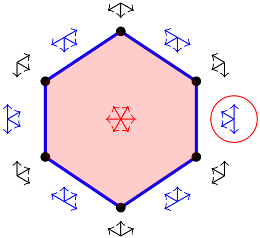






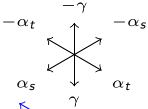
Equivalent definitions

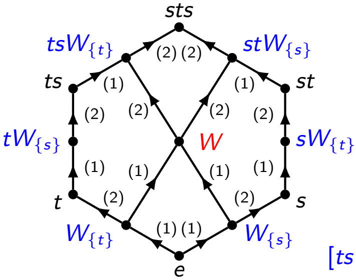
Theorem (D., Hohlweg, Pilaud [2016])

The following conditions are equivalent for two standard parabolic cosets xW_I and yW_J in the Coxeter complex $\mathcal{P}_W$

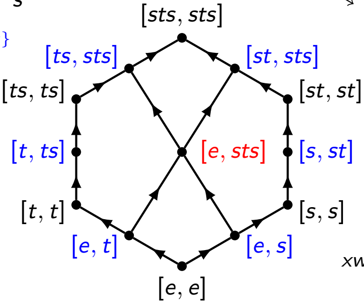
1. $xW_I \leq_F yW_J$
2. $\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \Phi^-$ *and* $\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \Phi^+$.
3. $x \leq_R y$ *and* $xw_{\circ,I} \leq_R yw_{\circ,J}$.

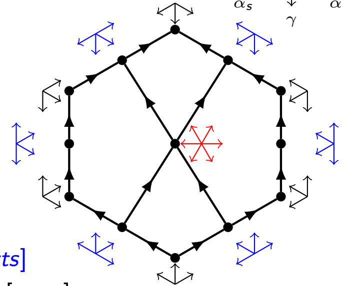
Equivalence for type A_2 Coxeter System





$xW_I \leq_F yW_J$





$\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \mathbf{\Phi}^-$
$\mathbf{R}(yW_J) \smallsetminus \mathbf{R}(xW_I) \subseteq \mathbf{\Phi}^+$

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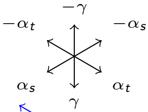

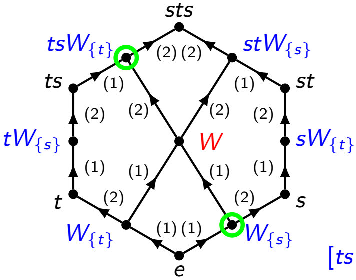

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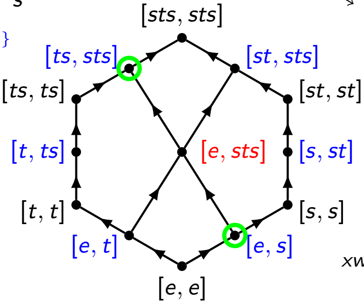

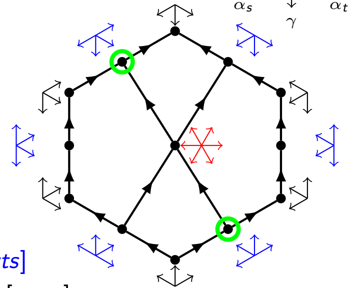

$\mathbf{R}(xW_I) \smallsetminus \mathbf{R}(yW_J) \subseteq \mathbf{\Phi}^-$

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$x \leq_R y$

$xw_{\circ,I} \leq_R yw_{\circ,J}$

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Facial weak order lattice

Theorem (D., Hohlweg, Pilaud [2016])

The facial weak order $(\mathcal{P}_W, \leq_F)$ is a lattice with the meet and join of two standard parabolic cosets xW_I and yW_J given by:

$$xW_I \wedge yW_J = z_\wedge W_{K_\wedge},$$
$$xW_I \vee yW_J = z_\vee W_{K_\vee}.$$

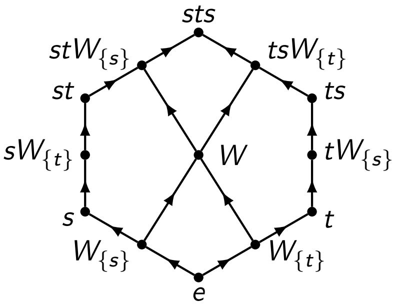
where,

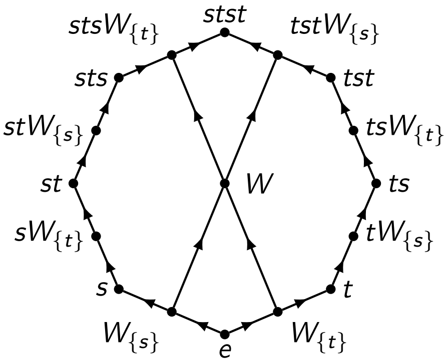
$$z_\wedge = x \wedge y \quad \text{and} \quad K_\wedge = D_L\big(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J})\big), \text{ and}$$
$$z_\vee = xw_{\circ,I} \vee yw_{\circ,J} \quad \text{and} \quad K_\vee = D_L\big(z_\vee^{-1}(x \vee y)\big)$$

Corollary (D., Hohlweg, Pilaud [2016])

The weak order is a sublattice of the facial weak order lattice.

Example: A_2 and B_2





Example: A_2 and B_2

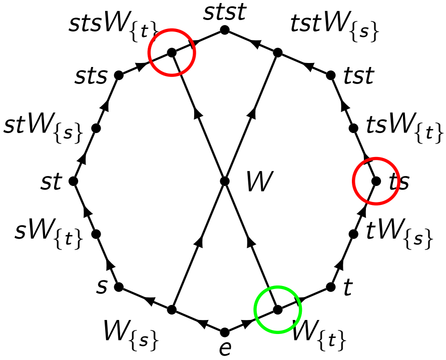
Example (Meet example)

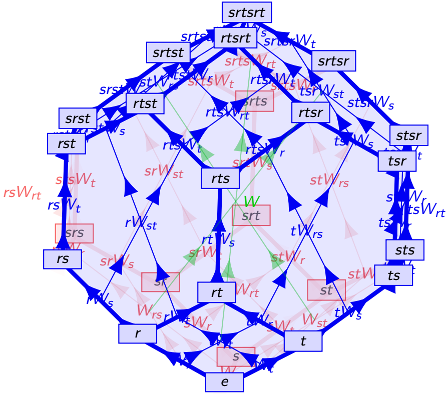
Recall

$$xW_I \wedge yW_J = z_\wedge W_{K_\wedge}$$
$$\text{where} \quad z_\wedge = x \wedge y$$
$$K_\wedge = D_L(z_\wedge^{-1}(xw_{\circ,I} \wedge yw_{\circ,J}))$$

We compute $ts \wedge stsW_{\{t\}}$.

$$z_\wedge = ts \wedge sts = e$$
$$K_\wedge = D_L(z_\wedge^{-1}(tsw_{\circ,\emptyset} \wedge stsw_{\circ,t}))$$
$$= D_L(e(ts \wedge stst))$$
$$= D_L(ts) = \{t\}.$$

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Thank you!